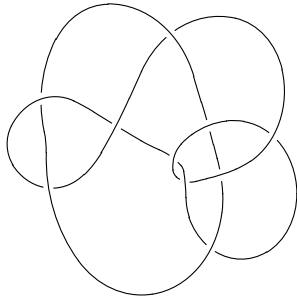


Notes on the Sigma invariants

Version 2

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$$\begin{aligned}
 r = & a^2 b^{-1} a^{-1} b a b a^{-2} b a b^{-2} a b a^{-2} b a b \cdot \\
 & a^{-1} b^{-1} a^2 b^{-1} a^{-1} b^{-1} a b a^{-2} b a b a^{-2} b \cdot \\
 & a b^{-1} a^{-1} b^{-1} a^2 b^{-1} a^{-1} b^2 a^{-1} b^{-1} a^2 b^{-1} \cdot \\
 & a^{-1} b^{-1} a b a^{-2} b a b a^{-1} b^{-1} a^2 b^{-1} a^{-1} b^{-1}
 \end{aligned}$$

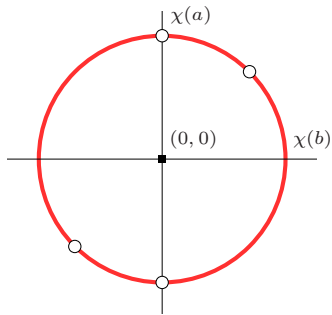
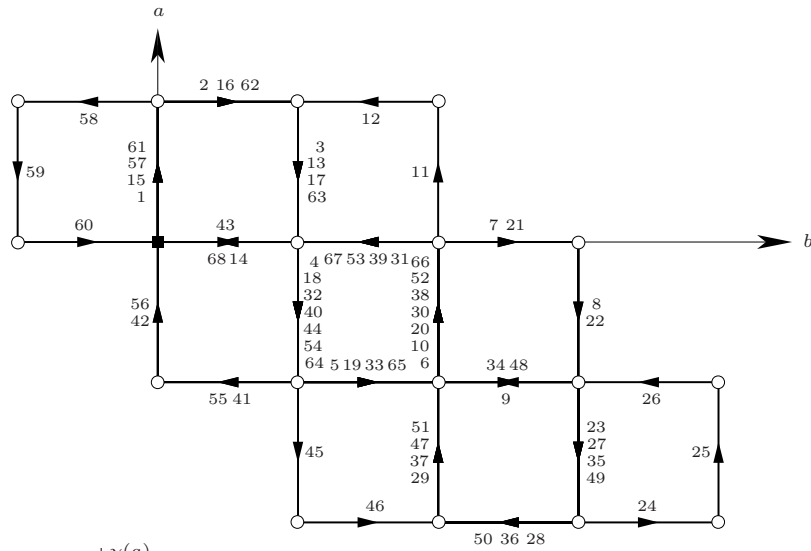


Diagram of the link L10n43, defining relator of its group,
path tracing out the relator and invariant of the group

Preface

Towards the end of the 1980s Robert Bieri and I started to write up a monograph on the Sigma invariants. The first such invariant had been introduced in 1980 in a paper that dealt mainly with soluble groups and culminated in a characterization of the *finitely related* metabelian groups among the *finitely generated* metabelian groups ([BS80]). Several years later, Robert Bieri, Walter Neumann and R. Strebel invented new Sigma invariants, defined for all finitely generated groups, and showed that they had implications far beyond the realm of soluble groups. The theory put forward in [BNS87] enables one, in particular, to characterize those normal subgroups N of a finitely generated group G that are finitely generated (as groups) and contain the derived group G' of G . At about the same time, Ken Brown, Gilbert Levitt, Gaël Meigniez and Jean-Claude Sikorav found alternate definitions for one of the Sigma invariants studied in [BNS87] (see [Bro87b], [Lev87], [Mei87] and [Sik87]). And a bit later, Robert Bieri and his student Burkhardt Renz succeeded in defining higher dimensional Sigma-invariants and in establishing analogues of some of the main theorems of [BNS87] (see [BR88] and [Ren89]). Among their results, is a characterization of the normal subgroups N of a finitely presented group G that are finitely presentable and contain G' .

1. By the end of the 1980s, time seemed ripe for a comprehensive account of the various kinds of Sigma invariants developed in the previous decade and of their mutual relations. Robert Bieri and I started work on such a memoir in 1988 and completed a first version in 1992. This version consisted of 4 chapters, written up in detail, and 3 appendices, the first one dealing with metabelian groups, the original motivation for the creation of the theory, the second one sketching the higher dimensional theory, the third one being a collection of notes. The first two of these appendices indicated that this version was only half-finished, and so we tried to complete it in the following years. But we did not reach this goal.

2. After my retirement in 2007, I made a new attempt at completing the monograph. My plan was to write a memoir with roughly the same scope, but to incorporate new developments that had taken place in the intervening years, and to present applications to other parts of Group Theory and attractive examples as early as feasible. These tasks turned out to be more time consuming than expected, and so I have decided to publish the monograph in several installments of which this is the second one.

3. In presenting these *Notes on the Sigma Invariants* it is a pleasure to acknowledge the help I received from many colleagues. My foremost thanks go to Robert Bieri. It is with him that I tried to fathom the structure of finitely presented metabelian groups in the second half of the 1970s, a problem that led us

in 1980 to introduce the first Sigma invariant. This introduction was followed in the 1980s by several joint papers dealing with various aspects of the invariant and also generalizations thereof, and culminated in 1987 with the introduction of the invariant $\Sigma_{G'}(G)$, this time in collaboration with Walter Neumann. My debt to Robert is also substantial in another respect: the first chapter of these *Notes* is in content and form close to Chapter I of the monograph [BS92] which, in turn, is an outgrowth of a lecture Robert gave in the second half of the 1980s.

I owe also many thanks to other colleagues. Their suggestions have served me well; I list their names in alphabetical order: Roger Alperin, Gilbert Baumslag, Markus Brodmann, Ken Brown, Gerhard Burde, Ian Chiswell, Laura Ciobanu, Yves de Cornulier, Reinhard Diestel, Nathan Dunfield, Martin J. Dunwoody, Ross Geoghegan, Slava Grigorchuk, John Groves, Pierre de la Harpe, Jim Howie, Manfred Karbe, Desi Kochloukova, Jürg Lehnert, Charles Livingston, John Meier, Chuck Miller III, Gaël Meigniez, Peter M. Neumann, Derek J. S. Robinson and Hanspeter Scherbel. To all of them, I express my gratitude.

4. Much of this text has been written in a small village in the Swiss Alps. I maintained contact with the world outside mostly through mails, but had the good luck of being invited to two stimulating meetings: the first one, the *Finitely Presented Solvable Groups Conference*, taking place in March 2011 at the City College of New York and being organized by Gilbert Baumslag, Stuart Margolis, Gretchen Ostheimer, Vladimir Shpilrain, and Sean Cleary. The second one, a workshop held at the Erwin Schrödinger International Institute in Vienna, organized by Goulmira Arzhantseva and Mark Sapir in December 2011 and entitled *Infinite Monster Groups*. I thank all the organizers for giving me these opportunities.

Almost all books contain errors, and these *Notes* will be no exception. I thus encourage readers to communicate to me misprints and errors, but also comments, criticism, questions and suggestions. Their contributions will be well-come.

Feldis, February 2013

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Introduction

The origin of the Sigma invariants lies in an answer to a question G. Baumslag had raised in 1973. In the early 1970s, G. Baumslag and V. N. Remeslennikov had discovered independently that there are many more finitely generated metabelian groups with a finite presentation than one might have suspected. They proved, in particular, that *every finitely generated metabelian group embeds into a finitely presented metabelian group* ([Bau73], [Rem73]). G. Baumslag therefore asked ([Bau74, p. 70]):

PROBLEM 1 *Is there any way of discerning finitely presented metabelian groups from the other finitely generated metabelian groups?*

Schur's multiplier $H_2(-, \mathbb{Z})$ is not sufficiently discriminating to settle this problem. This fact called for a search of additional conditions that a finitely presentable metabelian or, more generally, soluble group must satisfy. One such condition was published by R. Bieri and R. Strebel in 1978 (cf. [BS78, Thm. A]):

THEOREM 2 *Assume G is a soluble group and N is a normal subgroup with infinite cyclic quotient. Choose an element $t \in G$ that generates a complement of N . If G can be finitely presented, then G is an ascending HNN-extension over a finitely generated base group B contained in N .*

More precisely, there exists a sign ε so that $u = t^\varepsilon$ and B satisfy the conditions

$$B \subseteq uBu^{-1} \quad \text{and} \quad \bigcup_{j \geq 0} u^j Bu^{-j} = N. \quad (1)$$

Suppose now that G is a finitely presented soluble group whose abelianization $G_{\text{ab}} = G/G'$ has torsion-free rank $r_0(G_{\text{ab}})$ greater than 1. Then G contains infinitely many normal subgroups N to which Theorem 2 can be applied. To express the impact of all these applications, one needs a space made up of all possible couples (N, ε) and a subset of this space that records the couples for which requirement (1) is satisfied. Such a space is the rational sphere $S_{\mathbb{Q}}(G)$ associated to G and the subset recording the answers is a primitive version of an Sigma invariant. In the refined version, the rational sphere $S_{\mathbb{Q}}(G)$ is replaced by its ambient real sphere $S(G)$ and conditions (1) are supplemented by a new type of requirement which makes sense for every point of the sphere $S(G)$. This new condition was discovered around 1985 by Bieri, Neumann and Strebel; the answer to Problem 1, as given in [BS80], uses a weaker substitute.

Sigma invariants and their ambient sphere

I next describe some invariants that will play a rôle in this monograph, beginning with their ambient sphere.

The sphere $S(G)$

Common to all invariants denoted by Sigma is the fact that they associate to a group G a subset of a sphere $S(G)$. Typically, G is an infinite, finitely generated group; for certain invariants the group must satisfy more stringent conditions. The sphere $S(G)$ is derived from the real vector space $\text{Hom}(G, \mathbb{R})$; this vector space consists of all homomorphisms $\chi: G \rightarrow \mathbb{R}$ of G into the additive group of the field \mathbb{R} . In case G is the fundamental group of a path connected topological space X , $\text{Hom}(G, \mathbb{R})$ is canonically isomorphic to $\text{Hom}(H_1(X, \mathbb{Z}), \mathbb{R})$ and to $H^1(X, \mathbb{R})$.

Each non-zero homomorphism $\chi: G \rightarrow \mathbb{R}$ gives rise to a submonoid of G , namely

$$G_\chi = \{g \in G \mid \chi(g) \geq 0\} = \chi^{-1}([0, \infty)). \quad (2)$$

This submonoid does not change if χ is replaced by a positive multiple; the monoids are therefore parametrized by the open rays in $\text{Hom}(G, \mathbb{R})$ emanating from the origin. These rays are the points of the space $S(G)$, called *character sphere* of G . If G is finitely generated, the real vector space $\text{Hom}(G, \mathbb{R})$ is finite dimensional and carries a canonical topology induced by its norms. The space $S(G)$ equipped with the quotient topology is then homeomorphic to the unit sphere in a Euclidean vector space of the appropriate dimension.

Homotopical and homological invariants

There are two kinds of Sigma invariants. Those of the first sort depend only on a group G and are often called *homotopical*, partly because the definition of some of them is in terms of homotopical properties of a space associated to G , partly because they depend only on G , similar to the homotopy groups $\pi_n(X, x_0)$ which depend only on a pointed space (X, x_0) . The invariants of the other type depend on a couple (G, A) consisting of a finitely generated group G and a $\mathbb{Z}G$ -module A or, more generally, a G -operator group. These invariants are called *homological* for analogous reasons.

The invariant $\Sigma^0(G; A)$

The ancestor of all later invariants was introduced in [BS80] and denoted there by $\Sigma_A(G)$. It depends on a finitely generated *abelian* group G and a finitely generated $\mathbb{Z}G$ -module A ; it is thus of homological type. Its definition can be stated easily and shows how the submonoids enter into play; so I shall give it here in spite of the fact that it plays no rôle in this version of the monograph. I shall, however, use the notation proposed by R. Bieri and B. Renz in [BR88].

$$\Sigma^0(G; A) = \{[\chi] \in S(G) \mid A \text{ finitely generated over the monoid ring } \mathbb{Z}G_\chi\}. \quad (3)$$

In terms of this invariant, the main results of [BS80], Theorems A and B, can now be expressed as follows:

THEOREM 3 *Let H be a finitely generated group, $G = H_{\text{ab}}$ its abelianization and $A = H'/H''$ the abelianization of its derived group H' viewed as a left $\mathbb{Z}G$ -module via conjugation. Then A is a finitely generated $\mathbb{Z}G$ -module and the following statements hold:*

- (i) *if H is finitely related and soluble, then $\Sigma^0(G; A) \cup -\Sigma^0(G; A) = S(G)$;*
- (ii) *if H is metabelian and $\Sigma^0(G; A) \cup -\Sigma^0(G; A) = S(G)$ then H is finitely related.*

Here $-\Sigma$ denotes the image of Σ under the antipodal map $[\chi] \mapsto [-\chi]$ of $S(G)$.

Generalizing the invariant $\Sigma^0(G; A)$

In [BS80] the invariant $\Sigma^0(G; A)$ is only defined for finitely generated *modules* A over the group ring of a finitely generated *abelian* group G . Definition (3), however, admits of an obvious generalization to finitely generated modules over an arbitrary finitely generated group. This extension appeared in [Str81a]. More radical generalizations were then proposed in [BNS87].

These later generalizations are far from evident and so I would like to say a word on their motivation; to do so, I give some details of the proof of assertion (i) in Theorem 3. This assertion claims that the invariant $\Sigma^0(H_{\text{ab}}; H'_{\text{ab}})$ of a finitely related soluble group H must contain at least one point from every pair $\{[\chi], [-\chi]\}$ of antipodal points. The proof consists, roughly speaking, in representing the derived group H' of H as a free product with amalgamation $M_- \star_{M_0} M_+$, the representation depending on the point $[\chi] \in S(G)$. Since H' is soluble, this representation allows one to infer, after possible readjustment, that H' is either M_+ or M_- . An analysis of the generating systems of M_+ and M_- then leads to the conclusion that A is finitely generated either over $\mathbb{Z}G_\chi$ or over $\mathbb{Z}G_{-\chi}$.

The above outline explains the purpose of $\Sigma^0(G; A)$: it records local properties of the $\mathbb{Z}G$ -module A . In the proof of claim (ii) these local data are then used to find a finite set of relators of the metabelian group H . In this search, two properties of the sphere $S(G)$ and the invariant are exploited: the sphere is compact and $\Sigma^0(G; A)$ is an open subset of the sphere that comes equipped with a canonical open covering.

The method used in proving claim (ii) of Theorem 3 can be employed to obtain a result dealing with the module A . It states that A is finitely generated as an abelian group if, and only if, $\Sigma^0(G; A)$ coincides with $S(G)$. Actually a relative version of this result is also valid: instead of the entire sphere one considers great subspheres defined by

$$S(G, G_1) = \{[\chi] \in S(G) \mid \chi(G_1) = \{0\}\}. \quad (4)$$

Here G_1 is a subgroup of G . The relative version now reads:

$$S(G, G_1) \subseteq \Sigma^0(G; A) \iff A \text{ is finitely generated over the subring } \mathbb{Z}G_1. \quad (5)$$

The right hand side of this equation admits of a reformulation in simpler terms: let H_1 be the preimage of G_1 under the canonical projection $\pi: H \rightarrow G = H_{\text{ab}}$. Then the statement on the right holds precisely if the group H_1 is finitely generated.

The invariant $\Sigma(G)$

The characterization just stated is generalized in [BNS87] from finitely generated metabelian to arbitrary finitely generated groups. To do so, new invariants are introduced, two in the homotopical and two in the homological vein. Here I shall restrict attention to the most important among them; it is called $\Sigma(G)$ or, more explicitly $\Sigma_{G'}(G)$, in [BNS87] and defined like this:

$$\Sigma_{G'}(G) = \{[\chi] \in S(G) \mid G' \text{ is } fg \text{ over a } fg \text{ submonoid of } G_\chi\}. \quad (6)$$

The insistence on G' of being finitely generated, not over the entire, but over a finitely generated submonoid of G_χ , renders many of the proofs in [BNS87] technical and lengthy and may seem contrived. Notwithstanding this quibble, the invariant yields the characterization mentioned at the beginning of this section:

THEOREM 4 *Let N be a normal subgroup of the finitely generated group G with G/N abelian. Then N is finitely generated if, and, only if, $S(G, N) \subseteq \Sigma(G)$.*

In particular, G' is finitely generated if, and only if, $\Sigma(G) = S(G)$.

The question now arises whether there is a simpler, alternate definition of $\Sigma(G)$. The paper [BNS87] contains such a definition (stated in part (ii) of Proposition 3.4), but at the time of writing the paper none of the three authors seems to have noticed the geometric content of this new definition. Only later did Robert Bieri and, independently, Gaël Meigniez (see [Mei87, Section 6] or [Mei90, Theorem 3.19]) detect that this new definition can be restated by saying that a certain subgraph of the Cayley graph of G is connected.

The invariant $\Sigma^1(G)$

The invariant $\Sigma(G)$ when expressed in terms of Cayley graphs has been denoted by $\Sigma^1(G)$ since the late 1980s; the new definition can be stated as follows.

Let G be a finitely generated group, \mathcal{X} a finite generating set of G and let $\Gamma(G, \mathcal{X})$ be the Cayley graph of G with respect to the generating set \mathcal{X} . For each non-zero homomorphism $\chi: G \rightarrow \mathbb{R}$, let $\Gamma(G, \mathcal{X})_\chi$ denote the subgraph induced by the submonoid G_χ . Set

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \text{the subgraph } \Gamma(G, \mathcal{X})_\chi \text{ is connected}\}. \quad (7)$$

Then $\Sigma^1(G)$ does not depend on the choice of the finite generating set \mathcal{X} and it coincides with the invariant $\Sigma(G) = \Sigma_{G'}(G)$ of [BNS87].

The fact that $\Sigma^1(G)$ does not depend on \mathcal{X} reminds one of a similar fact underlying the definition, in terms of resolutions $\mathbf{F} \rightarrow \mathbb{Z}$, of the homology groups $H_*(G, \mathbb{Z})$ of the $\mathbb{Z}G$ -module \mathbb{Z} ; this fact can be exploited just as one does with resolutions: often a good choice of a generating set \mathcal{X} renders the computation of $\Sigma^1(G)$ easy.

So far, the definition of $[\chi] \in \Sigma^1(G)$ in terms of the connectedness of the subgraph $\Gamma(G, \mathcal{X})_\chi$ has been presented as a requirement that is simpler and more familiar than the condition stated in definition (6) of $\Sigma(G)$. But this is only one aspect of the new definition. More important is the discovery that several fundamental results about $\Sigma^1(G)$ — established in [BNS87] for $\Sigma(G)$ — can be proved for $\Sigma^1(G)$ by geometric arguments. These proofs have been found by Robert Bieri in the late 1980s. In addition, the definition of $\Sigma^1(G)$ in terms of connectedness of subgraphs of the Cayley graph indicates an avenue to analogous invariants $\Sigma^m(G)$ in higher dimensions: loosely speaking, the Cayley graph gets replaced by an m -dimensional space and connectivity by $(m-1)$ -connectivity. A main goal of these generalizations — carried out by Burkhardt Renz in his thesis [Ren88] under the supervision of Robert Bieri — is an analogue of Theorem 4 for higher dimensions. In it, the finite generation of the normal subgroup N is replaced by the property of being of type F_m . (A group N is of type F_m if there is an Eilenberg-MacLane space for N with finite m -skeleton.)

Contents of the actual version

This version comprises chapters A, B and C and a glimpse of two chapters that will be included in later versions.

Chapter A

The chapter begins with a few remarks about the sphere, associated to a finitely generated group, its geometrical structure and about Cayley graphs. The invariant Σ^1 itself is introduced in Section A2. In the following three sections, fundamental results about Σ^1 are established: first the so-called Σ^1 -criterion and the openness of Σ^1 in its ambient sphere. Theorem 4 is proved in the next section, while the final section A5 deals with a far reaching generalization of claim (i) in Theorem 3. The proofs in Chapter A use intensively the fact that Σ^1 is defined in terms of the connectivity of certain graphs and have a geometric ring.

Chapter B

In Chapter A, the emphasis is on basic properties of Σ^1 ; some examples, computations and applications are also given, but their goal is mainly to illustrate definitions and results. Chapter B supplements the first chapter by additional examples and detailed computations. It starts out with some words on the morphisms of spheres induced by homomorphisms $\varphi: G \rightarrow G_1$. These morphisms provide useful aids in computing the invariant of a group. As an illustration,

the invariant of graph groups (alias right angled Artin groups) is determined (see Theorem B1.17).

In Section B2 two algorithms are described that produce lower bounds for $\Sigma^1(G)$. The input for both algorithms is a set of relators satisfied by the chosen set of generators; typically, neither bound will coincide with $\Sigma^1(G)$. The algorithms are illustrated by examples of groups of piecewise linear homeomorphisms of the real line and by groups of links. The third section explores another theme: the sphere $S(G)$ of a finitely generated group G contains *points of rank 1*; they are represented by homomorphisms $\chi: G \rightarrow \mathbb{R}$ with infinite cyclic image. A point $[\chi]$ of rank 1 lies in $\Sigma^1(G)$ if, and only if, G is an *ascending HNN-extension with finitely generated base group* $B \subseteq N = \ker \chi$; see the statement of Theorem 2 for more details. The mentioned result is actually only one out of several consequences of a structure theorem for *finitely presented groups* G admitting a rank 1 character. In Section B3, this structure theorem is established and four of its consequences are discussed.

In the last section, an algorithm, due to Ken Brown [Bro87b], is derived; it allows one to compute the invariant of a group that is given by a presentation with a single defining relation.

Chapter C

One peculiarity of the invariant $\Sigma(G) = \Sigma_{G'}(G)$ was already noticeable in the account [BNS87]: $\Sigma(G)$ has alternate definitions which, at first sight, seem unrelated to it. Each such definition opens up new horizons; for this reason they are precious.

In Chapters A and B the Cayley graph definition is at center stage. Chapter C redresses the balance by giving voice to alternate definitions: they range from the main definition used in [BNS87], the description in terms of actions on \mathbb{R} -trees detected by Ken Brown [Bro87b], to definitions involving closed one-forms (cf. [Lev87]). Also established in Chapter C is the equality of Σ^1 with the invariant of homological type $\Sigma^1(-; \mathbb{Z})$ introduced by R. Bieri and B. Renz in [BR88].

Chapter D (preview)

In this chapter, three alternate definitions, introduced in Chapter C, will be studied in greater detail. The invariant $\Sigma^0(G; A)$ is the topic of the first two sections. In Section D1, we discuss results that are available for arbitrary finitely generated groups G and finitely generated G -modules A . If G is polycyclic, these results can be sharpened. In the second section, the group G is required to be (finitely generated) *abelian*; this is the original set-up of the invariant, introduced and exploited in [BS80]. In this special set-up, a new tool is available: the invariant $\Sigma^0(G; A)$ depends only on the commutative ring $R = \mathbb{Z}G / \text{Ann}_{\mathbb{Z}G}(A)$; moreover, characters $\chi: G \rightarrow \mathbb{R}$ representing points outside of $\Sigma^0(G; A)$ are characterized by the fact that they can be extended to valuations $v: R \rightarrow \mathbb{R} \cup \{\infty\}$ (see [BG84]).

In Section D3 we shall turn to the invariant $\Sigma_A(G)$ propounded in [BNS87] and establish results that have no direct counter-parts in the theory of Σ^1 and which allow one to deduce very useful properties of Σ^1 . In the final section D4, we shall give an algebraic characterization of transitive measured G -trees.

Chapter E (preview)

The invariant Σ^1 is defined for the class of all finitely generated groups, hence for 2^{\aleph_0} isomorphism types of groups. It encapsulates non-trivial information for each of these groups, provided, of course, one can compute the invariant with sufficient precision. Over the years, various classes of groups have been detected which are of interest to a wider segment of group theorists and for which Σ^1 can be computed.

One such class consists of graphs groups; its invariant is determined in section B1. Another one is formed by one-relator groups; their invariant can be worked out by an algorithm (see B4). In Chapter C, the invariants of two more classes are calculated, those of a sequence of groups introduced by C. H. Houghton in [Hou79] and those of wreath products.

The listed classes are by now means the only ones for which the computation of Σ^1 is possible and yields valuable insights. Chapter E will aim at presenting the most interesting of them.

A The invariant Σ^1

In this chapter, the first Σ -invariant is introduced and some of its properties are established. This invariant will be defined in terms of connectivity properties of subgraphs of Cayley graphs and denoted by Σ^1 . It has alternate definitions; some of them will be discussed in Chapter C.

A1 Setting the stage

This section sets the stage for the invariant $\Sigma^1(G)$ of a finitely generated group G . The invariant is a subset of a sphere $S(G)$ and consists of those points $[\chi]$ for which an associated subgraph Γ_χ of the Cayley graph $\Gamma(G, \mathcal{X})$ is connected.

Accordingly, the section starts out with the definition of the sphere $S(G)$, lists some of its properties and then turns to graphs. The invariant $\Sigma^1(G)$ will enter scene only in section A2.

A1.1 Characters and the character sphere

Let G be a finitely generated group. A homomorphism $\chi: G \rightarrow \mathbb{R}$ into the additive group of the field of real numbers will be called a *character of G* . The set $\text{Hom}(G, \mathbb{R})$ of all characters of G has the structure of a real vector space; as G is finitely generated it is finite dimensional. Its dimension is the subject of

LEMMA A1.1 *The dimension of $\text{Hom}(G, \mathbb{R})$ is equal to the torsion-free rank of the abelianization $G_{\text{ab}} = G/[G, G]$ of G .*

Proof. Since the additive group of \mathbb{R} is a torsion-free abelian group, every character $\chi: G \rightarrow \mathbb{R}$ factors over the canonical projection

$$\text{can}: G \twoheadrightarrow G_{\text{ab}} = G/[G, G] \twoheadrightarrow \overline{G} = G_{\text{ab}}/T(G_{\text{ab}}).$$

Here $T(G_{\text{ab}})$ denotes the *torsion-subgroup* of the abelianization G_{ab} of G ; it is finite. This projection induces an isomorphism of vector spaces

$$\text{can}^*: \text{Hom}(\overline{G}, \mathbb{R}) \xrightarrow{\sim} \text{Hom}(G, \mathbb{R}).$$

The quotient group \overline{G} , being a finitely generated torsion-free abelian group, is free abelian of rank n , say. Since a homomorphism $\bar{\chi}: \overline{G} \rightarrow \mathbb{R}$ is determined by its images on a basis of \overline{G} and as these images can be prescribed arbitrarily, the dimension $\dim_{\mathbb{R}} \text{Hom}(G, \mathbb{R})$ of the real vector space $\text{Hom}(G, \mathbb{R})$ equals the rank of

the free abelian group \overline{G} . This rank, in turn, coincides with the torsion-free rank $r_0(G_{\text{ab}})$ of G_{ab} . (For the justifications of the claims about abelian groups made in the above see, e. g., [Rob96, Section 4.2] or [LR04, Section 5.1].) \square

A1.1a Equivalent characters

Two characters χ_1, χ_2 of G will be called *equivalent* if there is a positive real number r with $\chi_1 = r \cdot \chi_2$. The equivalence class $[\chi]$ of a non-zero character χ is the ray emanating from 0 and passing through χ . The set of all equivalence classes

$$S(G) = \{[\chi] \mid \chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\}\} \quad (\text{A1.1})$$

together with the structure inherited from $\text{Hom}(G, \mathbb{R})$ will be called the *character sphere of the group G* .

REMARK A1.2 If the abelianized group G_{ab} is finite, the vector space $\text{Hom}(G, \mathbb{R})$ is reduced to the zero homomorphism and the character sphere $S(G)$ is empty. The methods and results of this monograph are of no interests for such groups.

In the remainder of section A1.1 some of its properties of the sphere $S(G)$ are discussed. Further features will be studied in section B1.1.

A1.1b Rank of a point

We begin with a stratification of the sphere. Each character $\chi: G \rightarrow \mathbb{R}$ has a *rank* given by the \mathbb{Z} -rank of the additive group $\chi(G) \subset \mathbb{R}$. Equivalent characters have the same rank, so one can speak of the rank of a point $[\chi] \in S(G)$; it will be denoted by $\text{rk}[\chi]$. The points of rank 1 are also called *rational*. A point is rational if and only if it can be represented by a character $\chi: G \rightarrow \mathbb{R}$ with $\chi(G) = \mathbb{Z}$. The set of all rational points of $S(G)$ is the sphere of rational characters, denoted $S_{\mathbb{Q}}(G)$; it is dense in $S(G)$ with respect to the topology defined below (see Lemma B3.24).

A1.1c Geometric structure

The sphere $S(G)$ inherits from the \mathbb{R} -linear structure of $\text{Hom}(G, \mathbb{R})$ the following geometric features:

- a *topology*: since $V = \text{Hom}(G, \mathbb{R})$ is a finite dimensional real vector space, all norms $\|\cdot\|$ on it induce the same topology. The sphere $S(G)$, being a quotient of the open subset $V \setminus \{0\}$, inherits the quotient topology. Equipped with this topology, it is homeomorphic to the unit sphere \mathbb{S}^{n-1} in the Euclidian space of dimension $n = r_0(G_{\text{ab}})$.
- a *family of subspheres*: every subset \mathcal{S} of $V = \text{Hom}(G, \mathbb{R})$ gives rise to a great subsphere $\{[\chi] \mid \chi(\mathcal{S}) = \{0\}\}$ of $S(G)$. Particularly important are the

Notes

subspaces consisting of a characters that vanish on a subgroup $H \leq G$. The corresponding subsphere will then be denoted by

$$S(G, H) = \{[\chi] \mid \chi(H) = \{0\}\}. \quad (\text{A1.2})$$

- a *family of sub-hemispheres*: every open (respectively closed) half space \mathcal{H} of $\text{Hom}(G, \mathbb{R})$ gives rise to an open (respectively closed) hemisphere

$$\{[\chi] \mid \chi \in \mathcal{H} \setminus 0\}.$$

A1.1d Introduction of coordinates

The sphere $S(G)$ and its topology have been defined without the use of coordinates. Occasionally, however, it is convenient to work with a Euclidean model of $S(G)$. In such a situation, coordinates will be introduced as follows.

Let \mathbb{E}^n denote the real vector space \mathbb{R}^n with standard basis (e_1, \dots, e_n) and equipped with the usual inner product $\langle -, - \rangle$, and let \mathbb{Z}^n denote the standard lattice in \mathbb{E}^n . Consider now a finitely generated group G the abelianization of which has torsion-free rank n . Then the groups $\overline{G} = G_{\text{ab}}/T(G_{\text{ab}})$ and \mathbb{Z}^n are isomorphic and every isomorphism $\iota: \overline{G} \xrightarrow{\sim} \mathbb{Z}^n$ induces an epimorphism $\vartheta = \iota \circ \text{can}: G \twoheadrightarrow \mathbb{Z}^n$. This epimorphism and the scalar product on \mathbb{E}^n give rise to an isomorphism $\vartheta^*: \mathbb{E}^n \xrightarrow{\sim} \text{Hom}(G, \mathbb{R})$ which sends the vector $v \in \mathbb{E}^n$ to the character $g \mapsto \langle v, \vartheta(g) \rangle$. Its restriction to the unit sphere of \mathbb{E}^n induces then a homeomorphism of spheres

$$\sigma(\vartheta): \mathbb{S}^{n-1} \xrightarrow{\sim} S(G). \quad (\text{A1.3})$$

It sends the unit vector $u \in \mathbb{S}^{n-1}$ to the point $[\chi_u] \in S(G)$ where $\chi_u: G \rightarrow \mathbb{R}$ denotes the character

$$\chi_u(g) = \langle u, \vartheta(g) \rangle. \quad (\text{A1.4})$$

A1.2 On graphs and Cayley graphs

The definition of $\Sigma^1(G)$ uses the Cayley graph $\Gamma(G, \mathcal{X})$ of the group G with respect to a finite generating system \mathcal{X} of G . For this reason, we continue with a few remarks on graphs and Cayley graphs.

A1.2a Terminology and notation used for graphs

Most of the graphs occurring in this monograph will be Cayley graphs of groups. The definition of such a graph uses *oriented* edges; but it does not furnish inverses of edges provided by the definition. On the other hand, the geometric definition of a path presupposes that an edge can be traversed in both directions of an oriented

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edge. If the inverses of the edges are not furnished by the construction of the graph, they will therefore have to be added later on. These facts have influenced the type of graph that will be favoured in the sequel: it is a graph equipped with positively oriented edges e^+ , but lacking inverses e^- of the given edges:

DEFINITION A1.3 An *oriented graph* Γ is given by a non-empty vertex set $V = V(\Gamma)$, an edge set $E = E(\Gamma)$, and two maps $\iota: E \rightarrow V$, $\tau: E \rightarrow V$ assigning to an edge e its origin $\iota(e)$ and its terminus $\tau(e)$, respectively.

Given an oriented graph (V, E, ι, τ) , one constructs a new graph having the same vertex set, but a larger edge set. Specifically, one introduces for each edge $e \in E$ a new element e^{-1} such that the assignment $e \mapsto e^{-1}$ is injective and the sets $E^- = \{e^{-1} \mid e \in E\}$ and E are disjoint. Then one extends the inversion $E \rightarrow E^-$, $e \mapsto e^{-1}$ and the endpoint maps ι, τ to the union $E^\pm = E \cup E^-$ by setting

$$(e^{-1})^{-1} = e, \quad \iota(e^{-1}) = \tau(e) \text{ and } \tau(e^{-1}) = \iota(e).$$

Figure A.1 illustrates the concepts of a combinatorial graph, an oriented graph and a graph resulting from an oriented graph by adding inverse edges.

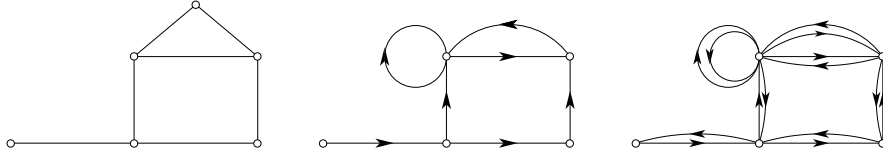


Figure A.1: A combinatorial, an oriented and a completed graph

An edge path of length $m \geq 1$ in Γ is a sequence of edges and inverse edges $p = e_1 e_2 \dots e_m$ such that $\tau(e_i) = \iota(e_{i+1})$ for all $i = 1, \dots, m-1$. The set of all edge paths of Γ will be denoted by $P(\Gamma)$; the set E^\pm can be thought of as a subset of $P(\Gamma)$. One extends the maps ι, τ and the inversion on E^\pm to $P(\Gamma)$ by putting $\iota(p) = \iota(e_1)$, $\tau(p) = \tau(e_m)$ and $p^{-1} = e_m^{-1} e_{m-1}^{-1} \dots e_1^{-1}$. It is also convenient to introduce for each vertex $v \in V$ a unique empty path $\emptyset_v \in P(\Gamma)$ with $\iota(\emptyset_v) = \tau(\emptyset_v) = v$ and $\emptyset_v^{-1} = \emptyset_v$. If one identifies $v \in V$ with $\emptyset_v \in P(\Gamma)$ one can consider the vertex set V as a subset of $P(\Gamma)$.

DEFINITION A1.4 A group G is said to *act on an oriented graph* $\Gamma = (V, E, \iota, \tau)$ if G acts on the sets V and E in such a way that the maps ι, τ are equivariant. By putting $g.(e^{-1}) = (g.e)^{-1}$, $g.(e_1 e_2 \dots e_m) = (g.e_1)(g.e_2) \dots (g.e_m)$ and $g.\emptyset_v = \emptyset_{g.v}$ the action of G on $V \cup E$ can be extended to a G -action on $P(\Gamma)$ so that $\iota(g.p) = g.\iota(p)$, $\tau(g.p) = g.\tau(p)$ and $(g.p)^{-1} = g.p^{-1}$ for all $g \in G$ and $p \in P(\Gamma)$.

If a is a vertex or an edge of the graph, the stabilizer $\{g \in G \mid g.a = a\}$ of a will be denoted by G_a . If all stabilizers are trivial, the group G is said to act *freely*.

Notes

A1.2b Terminology and notation used for Cayley graphs

Let G be a group. The construction of Cayley graph of G involves a generating set \mathcal{X} of G . If G/N is a quotient of G , the canonical image of \mathcal{X} on G/N generates, of course, the quotient group, but the map $\mathcal{X} \rightarrow G/N$ that sends x to xN may no longer be injective. In the sequel, we therefore use generating subsets only in special situations and work, in general, with maps $\eta: \mathcal{X} \rightarrow G$ the image of which generates G ; such maps will be called *generating systems* of G . To ease notation, the symbol η will often be suppressed.

DEFINITION A1.5 Let G be a group and $\eta: \mathcal{X} \rightarrow G$ a generating family of G . The *Cayley graph* $\Gamma = \Gamma(G, \mathcal{X})$ of G with respect to \mathcal{X} is the oriented graph

$$(G, G \times \mathcal{X}, \iota, \tau)$$

with vertices the elements of G , (positively) oriented edges the pairs $e = (g, x) \in G \times \mathcal{X}$, and origin and terminus given by $\iota(e) = g$ and $\tau(e) = g \cdot \eta(x)$.¹

The Cayley graph is equipped with a canonical G -action induced by left multiplication on the vertex set G and on the first factor of the edge set $G \times \mathcal{X}$.

The inverse edges of the Cayley graph form a set $G \times \mathcal{X}^{-1}$; here \mathcal{X}^{-1} denotes a set that is disjoint from \mathcal{X} and related to \mathcal{X} with a bijection $\text{inv}: x \mapsto x^{-1}$. Origin, terminus and inverse of edges and the inverse edges are then given by the same expressions: if $e = (g, y) \in G \times \mathcal{X}^{\pm}$ then $\iota(e) = g$, $\tau(e) = gy$ and $e^{-1} = (gy, y^{-1})$. (Of course, $(x^{-1})^{-1}$ is to be interpreted as x .)

An edge path $p = e_1 e_2 \cdots e_m \in P(\Gamma)$ is uniquely determined by its origin $\iota(p) \in G$ and the word $y_1 y_2 \cdots y_m \in W(\mathcal{X}^{\pm})$ with $e_i = (g_i, y_i)$. Thus the set of all paths $P(\Gamma)$ can be identified with the set of pairs $p = (g, w) \in G \times W(\mathcal{X}^{\pm})$; we shall use this identification whenever convenient. Using this notation, the origin, terminus and the inverse of p are then given by $\iota(p) = g$, $\tau(p) = g\eta(w)$ and $p^{-1} = (\tau(p), w^{-1})$. Finally, the action of the group G on the set of edge paths $P(\Gamma)$ is given by the left G -action on the first factor of $G \times W(\mathcal{X}^{\pm})$.

EXAMPLES A1.6 The Cayley graphs used in the monograph are those of infinite groups; as only a small patch of such a graph can be illustrated by a figure, the first two examples will be Cayley graphs of small finite groups.

a) The *dihedral group* D_4 of order 8 admits a presentation with generators a, t and defining relations $a^4 = 1, t^2 = 1, ta = a^3t$. The Cayley graph $\Gamma(D_4, \{a, t\})$ has 8 vertices and 16 (positively) oriented edges; see figure A.2.

b) The *quaternion group* Q_8 of order 8 is the subgroup of the multiplicative group of the field of quaternions \mathbb{H} . It is generated by the quaternions i and j , both elements of order 4. Its Cayley graph depicted on the right of figure A.2.

c) We now turn to Cayley graphs of infinite groups. The first one embeds into the plane, but the second one will no longer have this exceptional property. Let G

¹In the sequel, the product $g \cdot \eta(x)$ will often be written $g \cdot x$ or gx .

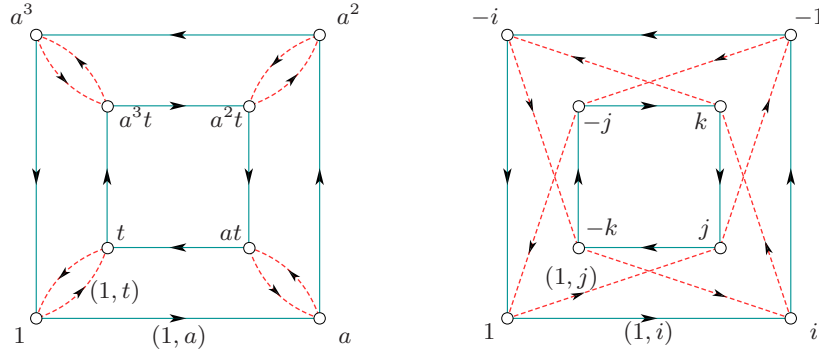


Figure A.2: Cayley graphs of the dihedral group and the quaternion group

be the *free abelian group of rank 2* and $\mathcal{X} = \{a, b\}$ a basis of G . If one identifies G with the standard lattice \mathbb{Z}^2 of the Euclidian plan \mathbb{R}^2 , the Cayley graph of G can be visualized by a Euclidean grid; see figure A.3.

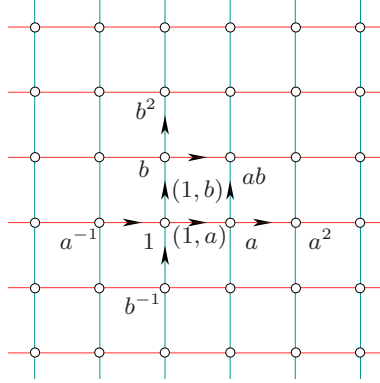


Figure A.3: Cayley graph of a free abelian group of rank 2

d) The Cayley graph of our last example is harder to visualize; in this respect, it is typical for the Cayley graphs of the groups that are at center stage in this monograph. Let $\mathbb{Z}[\frac{1}{2}]$ denote the additive group of all dyadic rational numbers acted on the left by an infinite cyclic group $C = \text{gp}(s)$, in such a way that the generator s acts by multiplication by 2, and let G to the semi-direct product $\mathbb{Z}[\frac{1}{2}] \rtimes C$. (This group is often referred to as Baumslag-Solitar group $BS(1, 2)$.) The product of two elements $(x, s^m), (x', s^{m'})$ of $\mathbb{Z}[\frac{1}{2}] \rtimes C$ is then given by the formula

$$(x, s^m) \cdot (x', s^{m'}) = (x + 2^m \cdot x', s^{m+m'}). \quad (\text{A1.5})$$

If one identifies C with the additive group of the ring \mathbb{Z} by means of the map

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$t^m \mapsto m$, the elements of G correspond to points in the plane; these points are dense on each horizontal line $\mathbb{R} \times \{m\}$ with $m \in \mathbb{Z}$.

The group G is generated by the elements $a = (1, s^0)$ and $u = (0, s)$. The Cayley graph Γ of G with respect to this generating set does not embed in the plane, but it is still useful to describe it in geometric terms. An edge $(g = (x, m), a)$ can be visualized by a segment of the horizontal line $\mathbb{R} \times \{m\}$ with origin $g = (x, m)$ and terminus $ga = (x + 2^m, m)$; note that its length is not constant, but increases with increasing second coordinate. An edge $(g = (x, m), u)$ corresponds to a segment of the vertical line $\{x\} \times \mathbb{R}$, starting at $g = (x, m)$ and ending in $(x, m + 1)$.

Our next aim is to convey an idea of the shape of the Cayley graph $\Gamma = \Gamma(G, \{a, u\})$. Consider the subset $M = \mathbb{Z} \times \{s^m \mid m \in \mathbb{N}\}$. The corresponding full subgraph has the form indicated by figure A.4. The Cayley graph is a union of such subgraphs. First of all, one has an infinite collection of subgraphs arising from the given one by translating it to the right by $x \in [0, 1) \cap \mathbb{Z}[\frac{1}{2}]$ units. The union of these subgraphs is a graph Γ_0 . This graph Γ_0 is contained in an isomorphic subgraph Γ_{-1} which arises from Γ_0 by rescaling the horizontal axis by $\frac{1}{2}$ and translating the rescaled graph by a unit towards the bottom. Similarly one defines a subgraph Γ_{-2} having the same relationship to Γ_{-1} , as Γ_{-1} has to Γ_0 . And so on and so forth.

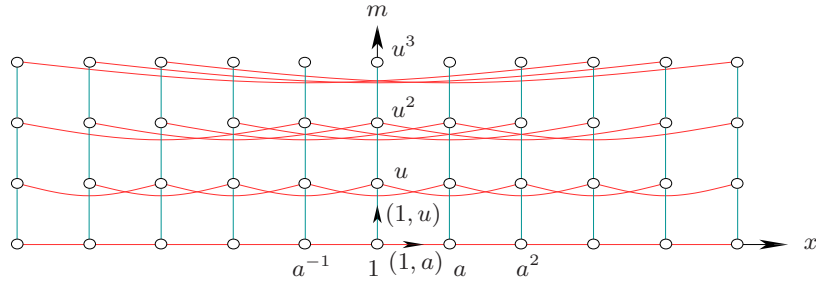


Figure A.4: A patch of the Cayley graph of the group defined by (A1.5)

A2 Enter the invariant Σ^1

In this section, the invariant is defined and computed in some simple cases.

A2.1 Definition

As in the previous section, G denotes a finitely generated group. Let $\eta: \mathcal{X} \rightarrow G$ be a finite system of generators of G and $\Gamma = \Gamma(G, \mathcal{X})$ the Cayley graph of G with respect to \mathcal{X} . For every character $\chi: G \rightarrow \mathbb{R}$ we define a subset

$$G_\chi = \{g \in G \mid \chi(g) \geq 0\}. \quad (\text{A2.1})$$

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This set is actually a submonoid of G ; we view it as a subset of the vertex set of the Cayley graph Γ and define $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$ to be the subgraph of Γ generated by the subset G_χ . Thus the vertex set of Γ_χ is the submonoid G_χ , the set of oriented edges consists of all the edges $(g, y) \in G \times \mathcal{X}$ with both $g \in G_\chi$ and $gy \in G_\chi$, and the incidence functions ι_*, τ_* are induced by the functions ι and τ of $\Gamma(G, \mathcal{X})$. If χ_1 and χ_2 are equivalent characters the subsets G_{χ_1} and G_{χ_2} coincide; so G_χ and Γ_χ depend only on the point $[\chi] \in S(G)$.

Since $\eta: \mathcal{X} \rightarrow G$ is a generating system, the graph $\Gamma(G, \mathcal{X})$ is connected. The subgraph Γ_χ , however, need not have this property. This prompts us to introduce the subset

$$\Sigma^1(G) = \{[\chi] \mid \Gamma_\chi(G, \mathcal{X}) \text{ is connected}\}; \quad (\text{A2.2})$$

it is often referred to as (*homotopical*) *geometric invariant of the group G* .

REMARKS A2.1 a) The definition of $\Sigma^1(G)$ involves the choice of a generating system \mathcal{X} , but the set $\Sigma^1(G)$ does not depend on this choice (see Theorem A2.3 below); in addition, it is invariant under isomorphisms (see section B1.2a). These facts justify the epithet *invariant*. The adjective *geometric* refers to the aspect that the mapping $G \mapsto \Sigma^1(G)$ associates to a (finitely generated) group G a geometric object, namely a subset of the sphere $S(G)$. In section A3, it will be proved that $\Sigma^1(G)$ is an open subset of $S(G)$.

b) The superscript 1 in the symbol Σ^1 is meant to indicate that Σ^1 is the first member in a sequence of invariants $\Sigma^2, \Sigma^3, \dots$; they are all subsets of the sphere $S(G)$, but are defined in terms of higher connectivity of higher dimensional spaces associated to groups.

A2.1a Examples illustrating the definition

In the following examples the global features of the Cayley graphs are so clear that the invariants can be determined by inspection.

1) Let G be the free abelian group of rank 2 and $\mathcal{X} = \{a, b\}$ a basis of G . If we identify G with the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ the Cayley graph with respect to the generating set \mathcal{X} can be visualized as a Euclidean grid. Every character $\chi: G \rightarrow \mathbb{R}$ extends to a unique \mathbb{R} -linear map $\chi_{\mathbb{R}}: \mathbb{R}^2 \rightarrow \mathbb{R}$; if $\chi \neq 0$, the subset G_χ is the intersection of the lattice \mathbb{Z}^2 with the half plane $\{x \in \mathbb{R}^2 \mid \chi_{\mathbb{R}}(x) \geq 0\}$. Figure A.5 indicates that Γ_χ is connected; a rigorous justification of this suggestion is easy and will be given later (see example A2.5a). So $\Sigma^1(G) = S(G)$.

2) The second example is the Baumslag-Solitar group $BS(1, 2)$ whose Cayley graph has been discussed in Example A1.6d). The group G is the semi-direct product of the abelian group $\mathbb{Z}[\frac{1}{2}]$ by an infinite cyclic group $C = \text{gp}(s)$, the generator s acting by multiplication by 2. The product of two elements of G is given by

$$(x, s^m) \cdot (x', s^{m'}) = (x + 2^m \cdot x', s^{m+m'}). \quad (\text{A2.3})$$

The group G is generated by the elements $a = (1, s^0)$ and $u = (0, s)$; they satisfy relation $uau^{-1} = a^2$. If one writes the relation in the form $a^{-1} \cdot uau^{-1} = a$, one

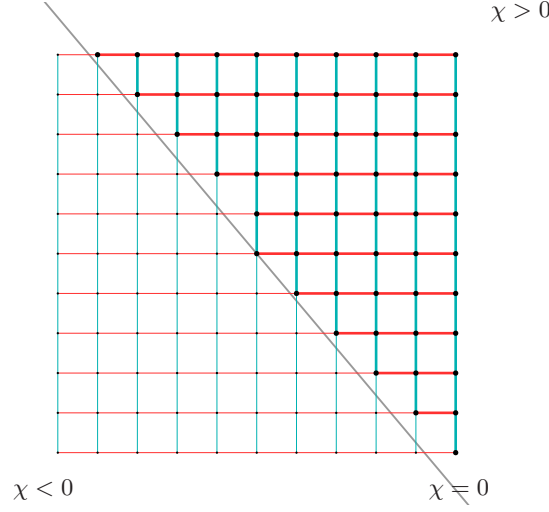


Figure A.5: Cutting in half the Cayley graph of a free abelian group of rank 2

sees that every character $\chi: G \rightarrow \mathbb{R}$ maps a to 0. So $S(G)$ is made up of only two points, the point $[\chi]$ with χ given by $\chi(u) = 1$ and its antipode $[-\chi]$.

The subgraph Γ_χ is a union of subgraphs isomorphic to the graph depicted in figure A.4. Each of these subgraphs is connected — actually they are the connected components of Γ_χ —, but Γ_χ itself is not connected.

The subgraph $\Gamma_{-\chi}$, however, is connected. To see this, note that for every number $x \in \mathbb{Z}[\frac{1}{2}]$ there exists a natural number n such that x is an integral multiple of 2^{-n} . The point $(x, -n)$ can therefore be connected to the origin by a path made up of a sequence of horizontal edges of length 2^{-n} followed by a sequence of n vertical edges. Since every vertex (x, m) with $m \leq 0$ can be connected to a vertex of the form $(x, -n)$ by a sequence of vertical edges, it follows that $\Gamma_{-\chi}$ is connected. Altogether we have shown that

$$S(G) = \{[\chi], [-\chi]\} \quad \text{and} \quad \Sigma^1(G) = \{[-\chi]\}. \quad (\text{A2.4})$$

3) We conclude with a class of groups whose invariants are empty. Let F_n be a free group of rank $n \geq 2$ or, more generally, a free product $G_1 \star G_2$ with non-trivial finitely generated factors G_1 and G_2 . If the abelianized group $G_{\text{ab}} \stackrel{\sim}{\leftarrow} (G_1)_{\text{ab}} \oplus (G_2)_{\text{ab}}$ is finite, $S(G)$ and hence $\Sigma^1(G)$ are empty. If not, let χ be non-zero character of G ; exchanging G_1 and G_2 if need be, we may assume that G_1 contains an element g_1 with $\chi(g_1) > 0$. Choose $g_2 \in G_2 \setminus \{1\}$ with $\chi(g_2) \geq 0$ and set $g = g_1^{-1}g_2g_1$. Then g is an element of G_χ whose unique normal form of g is (g_1^{-1}, g_2, g_1) . It follows that 1 and g cannot be connected inside $\Gamma(G, \mathcal{X})_\chi$ if $\eta: \mathcal{X} \rightarrow G$ is a suitably chosen finite generating system.

Indeed, let \mathcal{X} be the union $\mathcal{X}_1 \cup \mathcal{X}_2$ of a finite generating systems \mathcal{X}_1 of G_1 and \mathcal{X}_2 of G_2 . Consider a path $p = (1, w)$ from 1 to g . The word w is a product of

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subwords $w_1 \cdots w_k$ with each w_j either a word in \mathcal{X}_1^\pm or in \mathcal{X}_2^\pm . If one of these subwords represents $1 \in G$ the path p contains a loop; omitting it leads to a path p' containing a subset of the vertices lying along the path p . We may thus assume that none of the subwords w_j represents $1 \in G$. But if so, the uniqueness of the normal form implies that $\eta(w_1) = g_1^{-1}$; since $\chi(g_1^{-1})$ is negative the path p does therefore not run inside the subgraph $\Gamma_\chi(G; \mathcal{X}_1 \cup \mathcal{X}_2)$.

A2.2 Invariance

The definition of $\Sigma^1(G)$, as given by formula (A2.2), involves the generating system $\eta: \mathcal{X} \rightarrow G$. We shall prove that the set $\Sigma^1(G)$ does not depend on the choice of η and so deserves to be called an invariant of the group G .

In the proof of this claim and in many other verifications, an extension of a character χ from G to the set of all paths $P(\Gamma)$ on the Cayley graph $\Gamma = \Gamma(G, \mathcal{X})$ is useful. Given $\chi: G \rightarrow \mathbb{R}$ and a path $p = e_1 e_2 \cdots e_m = (g, y_1 y_2 \cdots y_m)$, set

$$\begin{aligned} v_\chi(p) &= \min\{\chi(\iota(e_1)), \chi(\tau(e_1)), \dots, \chi(\tau(e_m))\} \\ &= \min\{\chi(g), \chi(gy_1), \chi(gy_1y_2), \dots, \chi(gy_1 \cdots y_m)\} \\ &= \chi(g) + \min\{0, \chi(y_1), \chi(y_1y_2), \dots, \chi(y_1 \cdots y_m)\}. \end{aligned} \quad (\text{A2.5})$$

One verifies readily that v_χ satisfies the following identities:

$$\left. \begin{aligned} v_\chi(p^{-1}) &= v_\chi(p) && \text{for } p \in P(\Gamma), \\ v_\chi(g.p) &= \chi(g) + v_\chi(p) && \text{for } g \in G \text{ and } p \in P(\Gamma), \\ v_\chi(pq) &= \min\{v_\chi(p), v_\chi(q)\} && \text{for } p, q \in P(\Gamma) \text{ with } \tau(p) = \iota(q). \end{aligned} \right\} \quad (\text{A2.6})$$

REMARK A2.2 The set $W(\mathcal{X}^\pm)$ can be regarded as the set of edge path $p \in P(\Gamma)$ with $\iota(p) = 1$; so equation (A2.5) defines a map $v_\chi: W(\mathcal{X}^\pm) \rightarrow \mathbb{R}$. If the set $W(\mathcal{X}^\pm)$ is identified with a set of paths starting at 1, formula (A2.6) has to be adapted:

$$\left. \begin{aligned} v_\chi(w^{-1}) &= v_\chi(w) - \chi(w) && \text{for } w \in W(\mathcal{X}^\pm), \\ v_\chi(ww') &= \min\{v_\chi(w), \chi(w) + v_\chi(w')\} && \text{for } w, w' \in W(\mathcal{X}^\pm). \end{aligned} \right\} \quad (\text{A2.7})$$

THEOREM A2.3 *The subset $\Sigma^1(G) \subseteq S(G)$ does not depend on $\eta: \mathcal{X} \rightarrow G$.*

Proof. The passage from one finite generating system to another such system can be carried out in finitely many steps, each of which consists in adjoining or deleting a redundant generator. So it suffices to compare the Cayley graphs $\Gamma = \Gamma(G, \mathcal{X})$ and $\Gamma' = \Gamma(G, \mathcal{X} \cup \{z\})$. For every non-zero character $\chi: G \rightarrow \mathbb{R}$, the graphs Γ_χ and Γ'_χ have the same vertices and Γ_χ is a subgraph of Γ'_χ . Hence Γ'_χ is connected if Γ_χ is so.

Conversely, assume Γ'_χ is connected. Let $y_1 y_2 \cdots y_m$ be a word in \mathcal{X}^\pm which represents the redundant generator z . Next choose an element $t \in \mathcal{X}^\pm$ with

Notes

$\chi(t) > 0$ and a natural number k such that $\chi(t^k) = k\chi(t) \geq -v_\chi(y_1y_2 \cdots y_m)$. Given vertices g and h of Γ_χ , we can find a path from g to h that stays inside Γ_χ like this. Set $g' = t^{-k}gt^k$ and $h' = t^{-k}ht^k$. Then $\chi(g') = \chi(g)$ and so g' is a vertex of Γ_χ ; similarly, $h' \in \Gamma_\chi$. Because Γ'_χ is connected, there exist a path $p' = (g', w')$ inside Γ'_χ that leads from g' to h' . The word w' may contain the letter z or its inverse z^{-1} ; by replacing each occurrence of z or z^{-1} by the word $y_1y_2 \cdots y_m$, respectively its inverse, one obtains an \mathcal{X}^\pm -word w ; it gives rise to a path $p = (g', w)$ from g' to h' in the Cayley graph $\Gamma(G, \mathcal{X})$. This path need not be contained in the subgraph Γ_χ , but it can only leave it by an *a priori* known amount, for the construction of p guarantees that

$$v_\chi(p) \geq v_\chi(y_1y_2 \cdots y_m) \geq -\chi(t^k).$$

Let α_u denote the automorphism of the Cayley graph $\Gamma(G, \mathcal{X})$ that is induced by left multiplication by the element $u = t^k$. The translated path $\alpha_u(p)$ starts in $u \cdot g' = t^k \cdot (t^{-k}gt^k) = g \cdot t^k$, ends in $h \cdot t^k$ and runs inside Γ_χ , and so the path (g, t^kwt^{-k}) leads from g to h and stays inside Γ_χ . \square

A2.3 Consequences of the invariance

Theorem A2.3 is a basic result in the theory of the invariant Σ^1 ; it is also helpful in calculations in that it permits one to adapt the generating system to the situation at hand. The proofs of the following three results illustrate this aspect. ²

A2.3a Groups with non-trivial centre or with finitely generated derived group

PROPOSITION A2.4 *Let G be a finitely generated group with centre $\zeta(G)$. Then $\Sigma^1(G)$ contains every point $[\chi]$ which does not vanish on $\zeta(G)$. In symbols,*

$$S(G, \zeta(G))^c \subseteq \Sigma^1(G). \quad (\text{A2.8})$$

Proof. Let χ be a character with $\chi(\zeta(G)) \neq \{0\}$. Pick $z \in \zeta(G) \setminus \ker(\chi)$ and choose a generating set of G that includes z . Given any pair of elements g and h in G_χ there exists a path $p = (g, w)$ in $\Gamma(G, \mathcal{X})$ from g to h . If $v_\chi(p) \geq 0$, the path runs inside Γ_χ ; otherwise there exists a natural number k with $\chi(z^k) \geq |v_\chi(p)|$. The path $(g, z^k \cdot w \cdot z^{-k})$ then stays inside Γ_χ and it leads from g to $g \cdot z^k w z^{-k} = g \cdot w = h$. It follows that Γ_χ is connected and so $[\chi] \in \Sigma^1(G)$ by Theorem A2.3. \square

EXAMPLES A2.5 Here are two illustrations of Proposition A2.4.

a) Let G be a (finitely generated) *abelian* group. Then $\zeta(G) = G$ and so the set $S(G, \zeta(G))$ is empty. Proposition A2.4 therefore implies that $\Sigma^1(G) = S(G)$.

b) Let p, q be positive integers and G the group with generators x_1, x_2 and single defining relation $x_1^p = x_2^q$. The abelianization of G has \mathbb{Z} -rank 1 and so the

²Further results making use of Theorem A2.3 will be given in section B1.2.

sphere $S(G)$ consists of two points. These points are represented by the character $\chi_0: G \rightarrow \mathbb{R}$ with $\chi_0(x_1) = q$, $\chi_0(x_2) = p$ and its antipode. The element $z = x_1^p \in G$ commutes with the generator x_1 ; as it coincides with x_2^q , it commutes also with x_2 and so it belongs to the centre of the group G . Since neither χ_0 nor $-\chi_0$ map z to 0, Proposition A2.4 allows one to conclude that $\Sigma^1(G) = \{[\chi_0], [-\chi_0]\} = S(G)$.

PROPOSITION A2.6 *If the derived group of G is finitely generated $\Sigma^1(G) = S(G)$.*

Proof. Choose a finite generating system $\eta: \mathcal{X} \rightarrow G$ that includes a generating set \mathcal{Y} of G' . Given $g \in G_\chi$, represent g by an \mathcal{X} -word w . Reorder the letters in w so that those with positive χ -values precede the other letters, obtaining a word w' . Then the exponent sums of the word $w_1 = w \cdot (w')^{-1}$ with respect to the set of generators \mathcal{X} all vanish; so w_1 represents a element h in the derived group G' of G . Represent h by a \mathcal{Y}^\pm -word, say u . Then the path $p = (1, w' \cdot u)$ runs inside Γ_χ and links the unit element with g . \square

A2.3b Invariant of a direct product of groups

Given finitely generated groups G_1 and G_2 , let G denote their direct product $G_1 \times G_2$ and π_1, π_2 the canonical projections of $G_1 \times G_2$ onto G_i . If $\chi_1: G_1 \rightarrow \mathbb{R}$ is a non-zero character, the composition $\pi_1 \circ \chi_1$ is non-zero; so π_1 induces an embedding $\text{Hom}(\pi_1, \mathbb{1}): \text{Hom}(G_1, \mathbb{R}) \hookrightarrow \text{Hom}(G, \mathbb{R})$ of vector spaces and hence an embedding $\pi_1^*: S(G_1) \hookrightarrow S(G)$ of character spheres. Similarly, π_2 induces an embedding $\pi_2^*: S(G_2) \hookrightarrow S(G)$. These embeddings allow one to express the complement $\Sigma^1(G)^c$ of $\Sigma^1(G)$ by a simple formula:

PROPOSITION A2.7 *The complement of the invariant of a direct product $G_1 \times G_2$ of finitely generated groups G_1, G_2 is given by the formula*

$$\Sigma^1(G_1 \times G_2)^c = \pi_1^*(\Sigma^1(G_1))^c \cup \pi_2^*(\Sigma^1(G_2))^c. \quad (\text{A2.9})$$

Proof. Let \mathcal{X}_1 be a finite generating subset of G_1 which includes the neutral element 1_{G_1} , and let $\mathcal{X}_2 \subset G_2$ have the analogous properties. Then the set

$$\mathcal{X} = \mathcal{X}_1 \times \{1_{G_2}\} \cup \{1_{G_1}\} \times \mathcal{X}_2$$

is finite and it generates the direct product $G = G_1 \times G_2$. Theorem A2.3 allows one to base our computation on this generating set.

For each vertex $g = (g_1, g_2)$ of the Cayley graph $\Gamma(G, \mathcal{X})$ there exists an \mathcal{X}_1^\pm -word $y_{11}y_{12} \cdots y_{1h}$ representing the element g_1 and an \mathcal{X}_2^\pm -word $y_{21}y_{22} \cdots y_{2k}$ representing g_2 . These words give rise to the path

$$p_{12} = (1_G, (y_{11}, 1)(y_{12}, 1) \cdots (y_{1h}, 1) \cdot (1, y_{21})(1, y_{22}) \cdots (1, y_{2k})) \quad (\text{A2.10})$$

but also to the path

$$p_{21} = (1_G, (1, y_{21})(1, y_{22}) \cdots (1, y_{2k}) \cdot (y_{11}, 1)(y_{12}, 1) \cdots (y_{1h}, 1)). \quad (\text{A2.11})$$

Notes

Both paths connect the origin with the vertex (g_1, g_2)

Consider now a non-zero character $\chi: G \rightarrow \mathbb{R}$ and let $g = (g_1, g_2)$ be a vertex of the subgraph G_χ , i. e., an element of G with $\chi(g) \geq 0$. Assume first χ vanishes on G_2 and denote the restriction of χ to G_1 by χ_1 . Then a path of the form p_{12} runs inside the subgraph $\Gamma_\chi(G, \mathcal{X})$ if, and only if, the path $p_1 = (1, y_{11}y_{12} \cdots y_{1h})$ stays inside $\Gamma_{\chi_1}(G_1, \mathcal{X}_1)$. This shows that π_1^* maps $\Sigma^1(G_1)$ bijectively onto $\Sigma^1(G) \cap S(G, G_1)$. Similarly, π_2^* sends $\Sigma^1(G_2)$ bijectively onto $\Sigma^1(G) \cap S(G, G_2)$.

Assume now that χ vanishes neither on $G_1 \times \{1\}$ nor on $\{1\} \times G_2$. If $\chi(g_1, g_2) \geq 0$, then $\chi(g_1, 1) \geq 0$ or $\chi(1, g_2) \geq 0$. If $\chi(g_1, 1) \geq 0$, pick a path p_{12} from 1_G to g having the form (A2.10). This path may leave the subgraph Γ_χ , but its origin 1_G , its mid point $(g_1, 1)$ and its terminus g are in Γ_χ . We are going to modify the first half of the path, and later on the second one.

Let u be an element of $\{1\} \times \mathcal{X}_2^\pm$ with positive χ -value. For every $\ell \in \mathbb{N}$ the path

$$q_\ell = (1_G, u^\ell \cdot (y_{11}, 1)(y_{12}, 1) \cdots (y_{1h}, 1) \cdot u^{-\ell})$$

leads from the origin to the mid point $(g_1, 1)$ and it runs inside the subgraph Γ_χ if ℓ is large enough. The second part of the path p_{12} can be modified similarly. The result of both transformations is a path inside Γ_χ from 1_G to $g = (g_1, g_2)$.

If, on the other hand, $\chi(1, g_2) \geq 0$, one picks a path p_{21} from 1_G to g having the form (A2.11) and modifies it in like manner. \square

EXAMPLES A2.8 Proposition A2.7 allows one to work out the invariant of more complicated groups than those considered hitherto. Here are some specimens.

a) Let G_1 be the Baumslag-Solitar group discussed in example 2 of A2.1a; it has the presentation $\pi_1: \langle a_1, u_1 \mid u_1 a_1 u_1^{-1} = a_1^2 \rangle \xrightarrow{\sim} G_1$. Let G_2 be an isomorphic copy of G_1 , generated by elements a_2, u_2 , and set $G = G_1 \times G_2$. By the discussion carried out in example 2 of A2.1a the invariant $\Sigma^1(G_1)^c$ consists of the point $[\chi_1]$ with $\chi_1(u_1) = 1$ (and $\chi_1(a_1) = 0$); in view of Proposition A2.7 the complement $\Sigma^1(G)^c$ consists therefore of two points, the point $[\chi_1 \circ \pi_1]$ and the analogously defined point $[\chi_2 \circ \pi_2]$.

b) Next let $G = C \times F_2$ be the direct product of an infinite cyclic group C generated by t and a free group F_2 of rank 2 with basis $\{x_1, x_2\}$; the sphere of G is then two-dimensional. Since the element t lies in the centre of G , Proposition A2.4 implies that the complement $\Sigma^1(C \times F_2)^c$ is contained in the subsphere $S(G, C)$; this subsphere is a great circle that can be thought of as the equator of a 2-sphere. Proposition A2.7 and example 3 in A2.1a allow one to infer a stronger conclusion: $\Sigma^1(G)^c$ coincides with the equator $S(G, C)$.

c) Consider, finally, the direct product $G = F_2 \times F_2$ of two free groups of rank 2. Then G_{ab} is free abelian of rank 4 and so $S(G)$ is a 3-sphere. In view of Proposition A2.7 and example 3 in A2.1a, the complement $\Sigma^1(G)^c$ is the union of the subspheres $S(G, G_2)$ and $S(G, G_1)$. These subspheres are both great circles.

A2.4 The subspaces $\Gamma(G, \mathcal{X})_\chi^I$

We conclude this section with a comment on the definition of Σ^1 . Let $\Gamma(G, \mathcal{X})$ be the Cayley graph of a finitely generated group G , where $\eta: \mathcal{X} \rightarrow G$ is a finite generating family. Each non-zero character $\chi: G \rightarrow \mathbb{R}$ gives rise to a collection $I \mapsto G_\chi^I$ of subgraphs of Γ , one for each non-empty interval $I \subseteq \mathbb{R}$.

The collection is defined as follows: given a non-empty interval I , set

$$G_\chi^I = \{g \in G \mid \chi(g) \in I\}, \quad (\text{A2.12})$$

and then define Γ_χ^I to be the full subgraph of $\Gamma(G, \mathcal{X})$ generated by this subset. Special cases of this construction are $\Gamma_\chi = \Gamma_\chi^{[0, \infty)}$ and $\Gamma_{-\chi} = \Gamma_\chi^{(-\infty, 0]}$.

For every $g \in G$ the translated graph $g \cdot \Gamma_\chi^I$ coincides with $\Gamma_\chi^{\chi(g)+I}$; hence Γ_χ^I and Γ_χ^{r+I} are isomorphic graphs whenever $r \in \text{im } \chi$. It is not clear to what extent this conclusion remains valid for other real numbers r ; it does not hold in general, for G_χ^{r+I} may be empty if I is rather small. For a ray one has, however,

LEMMA A2.9 *If $\Gamma_{a_*} = \Gamma_\chi^{[a_*, \infty)}$ is connected for some $a_* \in \mathbb{R}$ then $\Gamma_a = \Gamma_\chi^{[a, \infty)}$ is connected for every $a \in \mathbb{R}$.*

Proof. Assume first $a < a_*$. Let $t \in \mathcal{X}^\pm$ be an element with positive χ -value and choose ℓ so large that $a + \chi(t^\ell) \geq a_*$. For every vertex $g \in \Gamma_a$ the path $p_g = (g, t^k)$ stays inside Γ_a and connects g with the vertex gt^ℓ that lies in Γ_{a_*} . This implies that Γ_a is connected. If, on the other hand, $a_* < a$, choose an element $h \in G$ with $a + \chi(h) < a_*$. Then the subgraph $h(\Gamma_a)$ is connected by the previous argument; as is isomorphic to Γ_a , this subgraph is connected, too. \square

The above lemma will find various applications in the sequel. A first one is

COROLLARY A2.10 *If $\Gamma(G, \mathcal{X})_\chi$ is not connected, it has infinitely many components.*

Proof. If Γ_χ has finitely many components, there exist a negative number a such that these components can be connected inside $\Gamma_\chi^{[a, \infty)}$. This implies that $\Gamma_\chi^{[a, \infty)}$ is connected, whence so is Γ_χ by lemma A2.9. \square

A3 The Σ^1 -criterion

Let G be a finitely generated group with finite generating system $\eta: \mathcal{X} \rightarrow G$. If G admits a non-zero character — and this is the only case we are genuinely interested in — the Cayley graph $\Gamma(G, \mathcal{X})$ is an infinite geometric object whose global form is, typically, hard to visualize. If a good grasp of $\Gamma(G, \mathcal{X})_\chi$ were a prerequisite for the determination of $\Sigma^1(G)$, one could not hope to work out the invariant, except in some simple cases. The Σ^1 -criterion now shows that a small number of

Notes

local conditions on the Cayley graph $\Gamma(G, \mathcal{X})$ imply that a subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected. In many examples of interest, this criterion enables one to determine $\Sigma^1(G)$ completely; in less favourable instances, it permits one to construct at least a non-empty, open subset of the invariant.³

This section divides into three parts: in the first, Theorem A3.1, the main result of the section, is stated and proved; it provides a necessary and sufficient condition for a point $[\chi]$ to belong to $\Sigma^1(G)$. Then two applications of the main result will be given; one of them says that $\Sigma^1(G)$ is an open subset of the sphere $S(G)$. In the third part, a variant of the Σ^1 -criterion is presented.

A3.1 The main result

It reads as follows:

THEOREM A3.1 *Let G be a group and $\eta: \mathcal{X} \rightarrow G$ a finite generating system of G . For every non-zero character $\chi: G \rightarrow \mathbb{R}$ and for every choice of $t \in \mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ with $\chi(t) > 0$, the following conditions are equivalent:*

- (i) *the subgraph $\Gamma(G, \mathcal{X})_\chi$ of the Cayley graph $\Gamma(G, \mathcal{X})$ is connected;*
- (ii) *for every $y \in \mathcal{Y}$ there exists a path p_y from t to $y \cdot t$ in $\Gamma(G, \mathcal{X})$ that satisfies the inequality $v_\chi(p_y) > v_\chi((1, y))$.*

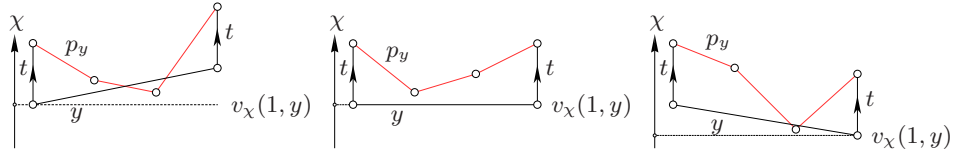


Figure A.6: Condition $v_\chi(p_y) > v_\chi((1, y))$ for $\chi(y)$ positive, zero or negative

REMARKS A3.2 a) Figure A.6 illustrates the condition $v_\chi(p_y) > v_\chi((1, y))$ which the edge paths p_y are required to satisfy. It reveals that the existence of the edge path $p_{y^{-1}}$ can be deduced from that of p_y .

b) Statement (ii) in Theorem A3.1 will be referred to as the *geometric version of the Σ^1 -criterion*. There exists more algebraic versions of the criterion; they will be discussed in section B2.

c) The geometric version of the Σ^1 -criterion is convenient in proofs of general results, examples being those of the main result or of Theorem A3.3 below. The algebraic versions are better suited to calculations of the invariant in examples.

d) The Σ^1 -criterion is due to Robert Bieri (cf. [BS92, Thm. 3.1]). It is the analogue of Proposition 2.1 given in [BNS87].

³The proof that a subgraph $\Gamma(G, \mathcal{X})_\chi$ is *not* connected requires often other techniques; they will be discussed in later chapters.

Proof. Assume first that statement (i) is valid. Given $y \in \mathcal{Y}$, define r_y to be the number $\min\{\chi(t), \chi(y \cdot t)\}$. Since Γ_χ is connected, the graph $\Gamma_\chi^{[r_y, \infty)}$ is connected by lemma A2.9. So there is an edge path $p_y = (t, w_y)$ from t to $y \cdot t$ with

$$v_\chi(p_y) \geq r_y = \min\{\chi(t), \chi(y \cdot t)\} = v_\chi((1, y)) + \chi(t) > v_\chi((1, y)).$$

By varying y we thus obtain a collection of paths p_y with $v_\chi(p_y) > v_\chi((1, y))$ and so statement (ii) holds.

Conversely, assume (ii) holds and define a number d by setting

$$d = \min \{v_\chi(p_y) - v_\chi((1, y)) \mid y \in \mathcal{Y}\};$$

it is positive. Now let $g \in G_\chi$ be a vertex of the subgraph Γ_χ ; we aim at constructing a path from 1 to g that does not leave this subgraph. Since the Cayley graph Γ is connected there exists a path p from 1 to g . This path p will now be subjected to the transformation $T: P(\Gamma) \rightarrow P(\Gamma)$ which replaces each edge (h, y) of p by the path (h, tw_yt^{-1}) and then eliminates the subpaths of the form $(h', t^{-1}t)$ (see Figure A.7). This transformation fixes the endpoints of p and satisfies the inequality

$$v_\chi(T(p)) \geq \min\{v_\chi(p) + d, 0\}. \quad (\text{A3.1})$$

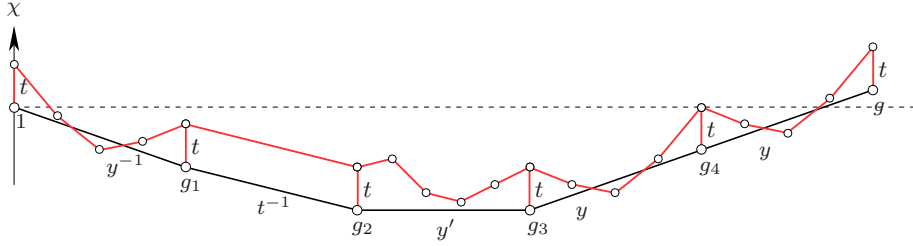


Figure A.7: The path transformation T

By iterating the described transformation, one obtains a sequence $p, T(p), T^2(p), \dots$ of paths from 1 to g that leave the subgraph Γ_χ by smaller and smaller amounts. If $\ell \geq |v_\chi(p)|/d$, the path $T^\ell(p)$ runs therefore inside Γ_χ from 1 to g . This shows that Γ_χ is connected. \square

A3.2 Corollaries of main result

We continue with two important consequences of Theorem A3.1.

A3.2a Openness of $\Sigma^1(G)$

The first consequence could be called a corollary of Theorem A3.1. In view of its importance we prefer to give it the title of a theorem.

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THEOREM A3.3 *For every finitely generated group G , the geometric invariant $\Sigma^1(G)$ is an open subset of the sphere $S(G)$.*

Proof. Suppose χ is a non-zero character of G that represents a point of $\Sigma^1(G)$. Let \mathcal{X} be a finite system of generators of G and choose an element $t \in \mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ with $\chi(t) > 0$. Implication (i) \Rightarrow (ii) of Theorem A3.1 then produces, for every $y \in \mathcal{Y}$, a path $p_y \in P(\Gamma(G, \mathcal{X}))$ from t to yt that satisfies the inequality $v_\chi(p_y) > v_\chi((1, y))$. As this is a *finite* system of inequalities it remains valid if χ is replaced by a character ψ sufficiently close to χ . Implication (ii) \Rightarrow (i) of Theorem A3.1 then applies to the character ψ and shows that $[\psi]$ belongs to $\Sigma^1(G)$. So the open neighbourhood

$$\mathcal{O}([\chi]) = \{[\psi] \mid \psi(t) > 0 \text{ and } v_\psi(p_y) > v_\psi((1, y)) \text{ for } y \in \mathcal{Y}\} \quad (\text{A3.2})$$

of $[\chi]$ is contained in $\Sigma^1(G)$. \square

The above theorem and the fact that closed subsets of the sphere $S(G)$ are compact, are key ingredients of some finiteness results that are amongst the main applications of the theory of Σ^1 . One such result is Theorem A4.1.

A3.2b Characterization of rank 1 points in Σ^1

A second consequence of Theorem A3.1 concerns rank 1 points in Σ^1 and describes them in terms of a widely known algebraic property of groups:

PROPOSITION A3.4 *If G is a finitely generated group, $\chi: G \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ a rank 1 character and $t \in G$ an element with $\chi(t) = 1$, the following conditions are equivalent:*

(i) $[\chi] \in \Sigma^1(G)$,

(ii) $N = \ker \chi$ contains a finitely generated group H with the properties

$$t^{-1}Ht \subseteq H \text{ and } \bigcup_{\ell \in \mathbb{N}} t^\ell H t^{-\ell} = N. \quad (\text{A3.3})$$

Proof. (i) \Rightarrow (ii). Since t generates a complement of $N = \ker \chi$, the group G admits a finite generating set of the form $\mathcal{X} = \{t\} \cup \{a_1, a_2, \dots, a_m\}$ where each generator a_j lies in N . As the subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected, implication (i) \Rightarrow (ii) of Theorem A3.1 then provides one, for $j = 1, 2, \dots, m$, with a path p_j that connects t with $a_j t$ and satisfies the inequality $v_\chi(p_j) > v_\chi((1, a_j)) = 0$; as χ takes values in \mathbb{Z} , this inequality amounts to say that $v_\chi(p_j) = 1$. Let w_j be the word with $p_j = (t, w_j)$. This word will then satisfy the relation $t^{-1}a_j t = w_j$ and the equality $v_\chi(w_j) = 0$. This equality can be restated by saying that w_j is freely equal to a product of conjugates $a_{i,\ell} = t^\ell a_i t^{-\ell}$ of the a_i with *non-negative* exponents ℓ .

on the Sigma invariants

Let μ denote the largest index ℓ for which some conjugate $a_{i,\ell}$ occurs in one of the words w_j and set

$$\mathcal{H} = \{a_{j,\ell} \mid 1 \leq j \leq m \text{ and } 0 \leq \ell \leq \mu\}.$$

Then \mathcal{H} generates a finitely generated subgroup H of N , and H satisfies the inclusion $t^{-1}Ht \subseteq H$. Indeed, if $\ell > 0$ and $a_{j,\ell} \in \mathcal{H}$ then $t^{-1}a_{j,\ell}t \in \mathcal{H}$ by the definition of the generators $a_{j,\ell}$ and of \mathcal{H} ; if $\ell = 0$, the relation $t^{-1}a_jt = w_j$ and the definition of \mathcal{H} guarantee that $t^{-1}a_{j,0}t \in H$. The statement $\bigcup_{\ell \in \mathbb{N}} t^\ell H t^{-\ell} = N$, finally, holds since N is generated by the conjugates $a_{j,\ell}$ with $1 \leq j \leq m$ and $\ell \in \mathbb{Z}$.

(ii) \Rightarrow (i). Let H be a finitely generated subgroup of N that enjoys properties (A3.3). Choose a finite set of generators $\mathcal{B} = \{b_1, b_2, \dots, b_f\}$ of H and express each conjugate $t^{-1}b_jt$ as a word w_j in \mathcal{B}^\pm ; such expressions exist by the assumption that $t^{-1}Ht \subseteq H$. In view of the second assumption in (A3.3) the set $\mathcal{X} = \{t\} \cup \mathcal{B}$ generates G . The Σ^1 -criterion is then satisfied for χ . Indeed, the relation $t^{-1}b_jt = w_j$ implies that the path $p_j = (t, w_j)$ leads from t to b_jt , and the fact that w_j is a word in \mathcal{B}^\pm and $\chi(\mathcal{B}) = \{0\}$ implies that

$$v_\chi(p_j) = v_\chi(w_j) + \chi(t) = 1 > 0 = v_\chi((1, b_j)).$$

Moreover, the path $p_t = (t, t)$ connects t with $t \cdot t$ and satisfies $v_\chi(p_t) > v_\chi((1, t))$. The claim now follows from Theorem A3.1. \square

REMARKS A3.5 a) Suppose H is a group that contains two isomorphic subgroups S and T . Each isomorphism $\mu: S \xrightarrow{\sim} T$ then leads to group, called *HNN-extension with base group H , associated subgroups S, T and stable letter y* , and defined by the presentation

$$G = \langle H, y \mid y \cdot s \cdot y^{-1} = \mu(s) \text{ for every } s \in S \rangle. \quad (\text{A3.4})$$

This construction was introduced and put to good use in the paper [HNN49] by G. Higman, B. H. Neumann and H. Neumann. Their analysis revealed, in particular, that the canonical homomorphism $\kappa: H \rightarrow G$ is injective. (See [LS01, Sect. IV.2] for more information on HNN-extensions.)

b) On comparing formulae (A3.3) and (A3.4), one sees that statement (ii) says that G is a HNN-extension with finitely generated base group H , associated subgroups $S = t^{-1}Ht$ and $T = H$ and stable letter t . The normal subgroup N is then the union of the ascending chain $H \subseteq {}^tH \subseteq {}^{t^2}H \subseteq \dots$ whence such an HNN-extension is often called *ascending*.

EXAMPLE A3.6 Given two relatively prime, non-zero integers p and q , consider the additive group $A = \mathbb{Z}[1/(p \cdot q)]$ of the subring of \mathbb{Q} generated by the rational number $1/(p \cdot q)$. Multiplication by the fraction p/q induces an automorphism $\mu = \mu_{p/q}$ on A and so one can form the semi-direct product $G = G_{p,q}$ of A and

Notes

an infinite cyclic group C generated by an element s that acts on A by μ . The multiplication of $G_{p,q}$ is then given by

$$(x, s^m) \cdot (x', s^{m'}) = (x + (p/q)^m \cdot x', s^{m+m'}).$$

The group $G_{p,q}$ is metabelian and a short calculation, based on Euclid's extended algorithm, shows that the elements $a = (1, s^0)$ and $t = (0, s)$ generate it.

If $p = q \in \{1, -1\}$, the group $G = G_{p,q}$ is free-Abelian of rank 2; in all other cases, the torsion-free rank of G_{ab} equals 1 and so $S(G)$ consists of two points; they are represented by the character χ that sends t to 1 and the subgroup A to $\{0\}$ and its antipode $-\chi$. If G is free-abelian, the invariant is the entire circle $S(G)$ (see example A2.5a). In the other cases, $\Sigma^1(G)$ can be determined with the help of Proposition A3.4. Note first that every finitely generated subgroup H of A is infinite cyclic, for such a group is contained in a group of the form $\mathbb{Z} \cdot (1/p \cdot q)^k$. So multiplication by the fraction $(p/q)^{-1} = q/p$ maps H into H if, and only if, q/p is an integer. It follows that $[\chi]$ lies in $\Sigma^1(G)$ if and only if $|p| = 1$ and that its antipode lies in $\Sigma^1(G)$ exactly if $|q| = 1$.

A3.3 An alternate definition of Σ^1

In Section A2, the subset Σ^1 has been defined in terms of the connectivity of subgraphs Γ_χ of the Cayley graph Γ . In this section, we investigate an alternate condition, involving the path components of a (sufficiently large) stripe of the Cayley graph, and show that it leads to the same invariant. The new definition will turn out to be very useful in the analysis, to be undertaken in section A5.1a, of a class of bi-partite graphs $\Delta_b(\chi)$.

A3.3a Comparing the components of $\Gamma(G, \mathcal{X})_\chi$

Let G be a finitely generated group, $\chi: G \rightarrow \mathbb{R}$ a non-zero character of G and Γ_χ the corresponding half of the Cayley graph of G (with respect to a finite generating system). If Γ_χ is not connected, it has infinitely many components (see Corollary A2.10). The next lemma implies that these components are isomorphic to each other.

LEMMA A3.7 *Let $\Gamma(G, \mathcal{X})$ be the Cayley graph of a group G with finite generating system $\eta: \mathcal{X} \rightarrow G$ and let χ be a non-zero character of G . If a and b are real numbers satisfying the conditions $a \leq b$ and $b \in \text{im } \chi$, the following assertions hold:*

- (i) *each component \mathcal{C} of the subgraph $\Gamma_\chi^{[a, \infty]}$ contains a vertex g with $\chi(g) = b$;*
- (ii) *the kernel of χ acts transitively on the components of $\Gamma_\chi^{[a, \infty)}$.*
- (iii) *the kernel of χ acts transitively on the components of $\Gamma_\chi^{(-\infty, -a]}$;*

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Proof. (i) It suffices to establish the claim for $b = 0$. Indeed, if $g \in G$ is an element with $\chi(g) = b$, the automorphism induced by g maps the subgraph $\Delta = \Gamma_\chi^{[a-b, \infty)}$ onto the subgraph $\Gamma_\chi^{[a, \infty)}$ and hence the components of the subgraph Δ onto those of $\Gamma_\chi^{[a, \infty)}$.

Consider a component \mathcal{C} of Δ . It contains a vertex, say h , with $\chi(h) \geq 0$. Choose an \mathcal{X}^\pm -word $w = y_1 y_2 \cdots y_k$ that represents h and reorder the letters of w so that the letters y_j with positive χ -values come before the other letters. If w' denotes the obtained word, the path $p = (1, w')$ runs in the subgraph Δ from 1 to a vertex h_1 with $\chi(h_1) = \chi(h)$. The latter condition implies that there exists an element $h_0 \in \ker \chi$ with $h = h_0 h_1$. The translated path $h_0.p = (h_0, w')$ then starts at h_0 and ends in h . The component \mathcal{C} , containing the endpoint h , contains therefore also $h_0 \in \ker \chi$.

(ii) Given components \mathcal{C} and \mathcal{C}' of $\Gamma_\chi^{[a, \infty)}$, there exist, by claim (i), vertices $g \in \mathcal{C}$ and $g' \in \mathcal{C}'$ with $\chi(g) = \chi(g') = b$. Then $h = g' \cdot g^{-1}$ lies in $\ker \chi$ and the automorphism of $\Gamma(G, \mathcal{X})$ induced by h sends g onto g' and hence \mathcal{C} onto \mathcal{C}' . This reasoning shows that $\ker \chi$ acts transitively on the components of $\Gamma_\chi^{[a, \infty)}$.

(iii) Statement (ii) holds for every non-zero character ψ , in particular for $-\chi$. Now use that $\Gamma_\chi^{(-\infty, -a]} = \Gamma_{-\chi}^{[a, \infty)}$. \square

A3.3b The new definition

The new definition is provided by statement (i) in

THEOREM A3.8 *Let G be a group generated by the finite system $\eta: \mathcal{X} \rightarrow G$ and let $\chi: G \rightarrow \mathbb{R}$ be a non-zero character. Given a real number a , let \mathcal{C} be one of the connected components of $\Gamma_\chi^{[a, \infty)}$. Then the following statements are equivalent:*

(i) *there is a real number $e \geq \max\{|\chi(x)| \mid x \in \mathcal{X}\}$ such that the intersection $\mathcal{C} \cap \Gamma_\chi^{[a, a+e]}$ is connected;*

(ii) $-\chi \in \Sigma^1(G)$.

Proof. Assume first statement (i) is valid and set $b = a + e$. Since e is at least as large as $\max\{|\chi(x)| \mid x \in \mathcal{X}\}$, every vertex of $\Gamma = \Gamma(G, \mathcal{X})$ lies in at least one of the subgraphs $\Gamma'_b = \Gamma_\chi^{(-\infty, b]}$ and $\Gamma_a = \Gamma_\chi^{[a, \infty)}$. This fact allows one to prove that Γ'_b is connected. Indeed, given two vertices g_1 and g_2 of Γ'_b , there exists a path p in Γ from g_1 to g_2 . The path p has a decomposition $p = p_1 p_2 \cdots p_m$ where each subpath p_i is either a path of Γ_a or a path of Γ'_b , and each intermediate endpoint lies in $\Gamma_\chi^{[a, b]}$. Let J denote the set of indices of the subpaths running in Γ_a . For every index $j \in J$, the subpath p_j runs in a component \mathcal{C}_j of Γ_a . This component \mathcal{C}_j is the image of the component \mathcal{C} under some graph automorphism induced by an element $h_j \in \ker \chi$ (see Lemma A3.7). The intersection $\mathcal{C}_j \cap \Gamma_\chi^{[a, b]}$ is therefore connected, for it is isomorphic to the intersection $\mathcal{C} \cap \Gamma_\chi^{[a, b]}$ which is connected by

Notes

statement (i). So there exists a path p'_j in the slice $\Gamma_\chi^{[a,b]}$ with the same endpoints as p_j . Upon replacing the subpaths p_j by the paths p'_j one obtains a path that runs in the subgraph Γ_b and connects the given vertices g_1 and g_2 of Γ_b . This proves that $\Gamma_b = \Gamma_\chi^{(-\infty, b]}$ is connected, whence $\Gamma_{-\chi} = \Gamma_\chi^{(-\infty, 0]}$ is so by Lemma A2.9 and thus $[-\chi]$ is a point of $\Sigma^1(G)$.

Conversely, assume that $[-\chi] \in \Sigma^1(G)$. We aim at imitating the proof of implication (ii) \Rightarrow (i) of Theorem A3.1. To bring out the similarity, we set $\psi = -\chi$ and argue in terms of this new character. By hypothesis, $[\psi] \in \Sigma^1(G)$. Choose an element $t \in \mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ with $\psi(t) > 0$ and use implication (i) \Rightarrow (ii) of Theorem A3.1 to find, for every $y \in \mathcal{Y}$, an edge path $p_y = (h, w_y) \in P(\Gamma)$ from t to yt with $v_\psi(p_y) > v_\psi(1, y)$. Let e be the larger of the two numbers

$$\begin{aligned} \max\{|\psi(x)| \mid x \in \mathcal{X}\} &= \max\{|\chi(x)| \mid x \in \mathcal{X}\}, \\ \max\{\psi(g) \mid g \text{ a vertex of } p_y \text{ for some } y \in \mathcal{Y}\} \end{aligned}$$

and set $b = a + e$. Suppose now that g_1 and g_2 are vertices of the slice $\mathcal{S} = \Gamma_\psi^{[-b, -a]}$ which lie in a component \mathcal{C} of $\Gamma_\psi^{(-\infty, -a]} = \Gamma_\chi^{[a, \infty]}$. Then there exists a path p in \mathcal{C} from g_1 to g_2 . We submit it to an endpoint preserving transformation $T': P(\Gamma) \rightarrow P(\Gamma)$ that is similar to the transformation T used in the proof of implication (ii) \Rightarrow (i) of Theorem A3.1. Two cases arise. If $v_\psi(p) \geq -b$, the path p itself runs in the slice \mathcal{S} and the transformation T' is defined to leave p as it is. Otherwise, set

$$d = \min\{v_\psi(p_y) - v_\psi(1, y) \mid Y \in \mathcal{Y}^\pm\}$$

and consider the edges (h, y) of p which are not contained in the slice \mathcal{S} . Replace each such edge by the path (h, tw_yt^{-1}) and delete afterwards subpaths of the form $(h', t^{-1}t)$. The resulting path $T'(p)$ will then run inside the slice $\Gamma_\chi^{[c_1, -a]}$ with $c_1 = \min\{-b, v_\psi(p) + d\}$; indeed, if h_0 is a vertex of p below the slice \mathcal{S} both the edge of p that terminates in h_0 and the following one are replaced. By iterating the transformation T' , one will then end up with a path in the slice $\mathcal{S} = \Gamma_\chi^{[a, b]}$ which links the given vertices g_1 and g_2 of \mathcal{S} . It follows that the intersection $\mathcal{C} \cap \Gamma_\chi^{[a, b]}$ is connected. \square

Theorem A3.8 has a pleasing corollary that deserves to be stated at this point, in spite of the fact that its significance will only become clear in section A4.1a.

COROLLARY A3.9 *Let G be a group generated by the finite system $\eta: \mathcal{X} \rightarrow G$ and let $\chi: G \rightarrow \mathbb{R}$ denote a non-zero character. For every real number a the following conditions are equivalent:*

- (i) *there is a real number $b \geq a + \max\{|\chi(x)| \mid x \in \mathcal{X}\}$ such that the slice $\Gamma_\chi^{[a, b]}$ is connected;*
- (ii) *$\Sigma^1(G, \mathcal{X})$ contains the pair of antipodal points $\{[\chi], -[\chi]\}$.*

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Proof. Assume first the slice $\Gamma_\chi^{[a,b]}$ is connected for some width $b-a \geq \max\{|\chi(x)|\}$. Thanks to the width of the slice every point in $\Gamma_\chi^{[a,\infty)}$ can be connected inside $\Gamma_\chi^{[a,\infty)}$ to a point of the slice, and so $\Gamma_\chi^{[a,\infty)}$ is connected. Similarly one sees that $\Gamma_\chi^{(\infty,b]}$ is connected. Lemma A2.9 then implies that the subgraphs Γ_χ and $\Gamma_{-\chi}$ are connected, whence $[\chi]$ and $[-\chi]$ belong to $\Sigma^1(G)$.

Conversely, if this conclusion holds the subgraphs $\Gamma_{-\chi}^{[-a,\infty)} = \Gamma_\chi^{(-\infty,a]}$ and $\Gamma_\chi^{[a,\infty)}$ are connected. Since $\Gamma_\chi^{(-\infty,a]}$ is connected, implication (ii) \Rightarrow (i) of Theorem A3.8 applies and provides us with a width $b-a \geq \max\{|\chi(x)|\}$ such that the intersection of the slice $\Gamma_\chi^{[a,b]}$ with a fixed connected component \mathcal{C} of $\Gamma_\chi^{[a,\infty)}$ is connected. As $\Gamma_\chi^{[a,\infty)}$ itself is connected, this says that the entire slice is connected. \square

A3.3c A characterization of descending HNN-extensions

Upon combining Proposition A3.4 and Theorem A3.8, one arrives at the following characterization of *descending* HNN-extensions:

PROPOSITION A3.10 *Let G be a group generated by the finite system $\eta: \mathcal{X} \rightarrow G$, let $\chi: G \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ be a rank 1 character and let $u \in G$ be an element with $\chi(u) = 1$. Finally, let \mathcal{C} denote the connected component of Γ_χ that contains the vertex 1_G . Then the following conditions are equivalent:*

- (i) *the intersection $\mathcal{C} \cap \Gamma_\chi^{[0,k]}$ is connected for some integer $k \geq 0$;*
- (iii) *$N = \ker \chi$ contains a finitely generated subgroup H with the properties*

$$uHu^{-1} \subseteq H \text{ and } \bigcup_{\ell \in \mathbb{N}} u^{-\ell}Hu^\ell = N. \quad (\text{A3.5})$$

The preceeding proposition can be illustrated neatly by the Baumslag-Solitar group G considered in example 2 of section A2.1a. The group is generated by elements $a = (1, 0)$ and $u = (0, 1)$, and it admits the presentation $\langle a, u \mid uau^{-1} = a^2 \rangle$, describing it is a descending HNN-extension with stable letter u . The following portion of the Cayley graph of G (cf. Figure A.4 on page 7) then shows that condition (i) holds for every $k \geq 0$.

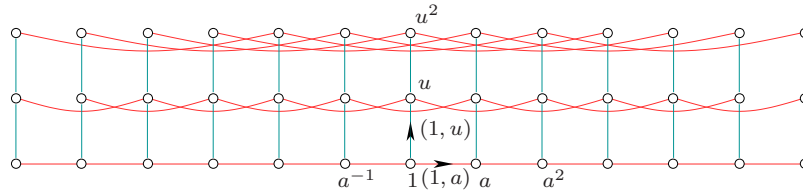


Figure A.8: A portion of the Cayley graph of G

A4 Finitely generated normal subgroups

In this section, we present one of the main results about the invariant Σ^1 . It deals with normal subgroups N of G containing the commutator subgroup $G' = [G, G]$ of G and gives an answer to the question whether such a subgroup is a finitely generated.

THEOREM A4.1 *Let G be a finitely generated group and $N \triangleleft G$ a normal subgroup with abelian factor group. Then the biimplication*

$$N \text{ is finitely generated} \iff S(G, N) \subseteq \Sigma^1(G) \quad (\text{A4.1})$$

holds. In particular, the commutator subgroup G' of G is finitely generated if $\Sigma^1(G) = S(G)$, and conversely.

The proof of implication \Rightarrow is fairly easy, as is the proof in the case where G/N is infinite cyclic (see Corollary A4.3); these two proofs are given in a preliminary discussion. The justification of implication \Leftarrow occupies the second section. In the final section, an application of Theorem A4.1 will be presented.

NOTE A4.2 Theorem A4.1 is the analogue of Theorem B1 in [BNS87]. The proof given in section A4.2 is an amplified version of an argument due to Robert Bieri (cf. [BS92, Thm. 4.1]).

A4.1 Preliminary investigation

As before, G denotes a finitely generated group, $N \triangleleft G$ a normal subgroup of G and $\pi: G \rightarrow Q$ the canonical projection of G onto the factor group $Q = G/N$.

A4.1a Normal subgroups with infinite cyclic factor group

If the factor group Q is infinite cyclic, N is the kernel of a rank 1 homomorphism $\chi: G \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ and Theorem A4.1 takes on the following simple form:

COROLLARY A4.3 *The kernel of a rank 1 character $\chi: G \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ is finitely generated if, and only if, $\{[\chi], [-\chi]\} \in \Sigma^1(G)$.*

The corollary admits proofs that are far simpler than the general proof. One of them can be based on Corollary A3.9 and Proposition A4.7 below. A second one follows from the characterization of rank 1 points afforded by Proposition A3.4. Here are the details:

Proof. Let $t \in G$ be an element with $\chi(t) = 1$. Assume first, $N = \ker \chi$ is finitely generated. Then the inclusions $t^{-1}Nt \subseteq N$ and $tNt^{-1} \subseteq N$ hold, and so G is an ascending HNN-extension with respect to t and with respect to t^{-1} ; by Proposition A3.4 both $[\chi]$ and $[-\chi]$ belong therefore to $\Sigma^1(G)$.

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Conversely, if $[\chi]$ and $[-\chi]$ lie in $\Sigma^1(G)$ then G is an ascending HNN-extension with finitely generated base group B_1 and stable letter t , say, and an ascending HNN-extension with finitely generated base group B_2 , say, and stable letter t^{-1} . So $t^{-1}B_1t \subseteq B_1$ and $tB_2t^{-1} \subseteq B_2$. Since B_1 is finitely generated and $\bigcup_{j \in \mathbb{N}} t^{-j}B_2t^j = \ker \chi$, the base group B_1 is contained in a conjugate of B_2 , say $B_1 \subseteq t^{-k}B_2t^k$. The chain of inclusions

$$t^\ell B_1 t^{-\ell} \subseteq t^\ell (t^{-k} B_2 t^k) t^{-\ell} \subseteq t^{-k} (t^\ell B_2 t^{-\ell}) t^k \subseteq t^{-k} B_2 t^k$$

holds therefore for every positive integer ℓ . As $\bigcup_{\ell \in \mathbb{N}} u^\ell B_1 u^{-\ell} = \ker \chi$ these inclusions show that the subgroup $u^{-k} B_2 u^k$ coincides with $\ker \chi$. But if so, $\ker \chi = B_2$ and thus $\ker \chi$ is finitely generated. \square

REMARK A4.4 The above proof makes use of Proposition A3.4 and general properties of ascending HNN-extensions. It gives no bound on the number of generators of the normal subgroup $N = \ker \chi$. An upper bound can be obtained by going back to the proof of implication (i) \Rightarrow (ii) of that proposition. Here are the details: let $t \in G$ be an element generating a complement of $N = \ker \chi$. Choose a finite subset $\mathcal{A} = \{a_1, \dots, a_m\}$ of N such that $\mathcal{X} = \mathcal{A} \cup \{t\}$ generates G . Then N will be generated by the conjugates $a_{j,\ell} = t^\ell a_j t^{-\ell}$ of the $a_j \in \mathcal{A}$.

Assume now that $[\chi]$ and $[-\chi]$ lie in $\Sigma^1(G)$. The hypothesis that $[\chi] \in \Sigma^1(G)$ and implication (i) \Rightarrow (ii) in Theorem A3.1 then provide us with words w_j such that

$$t^{-1} \cdot a_j \cdot t = w_j(\mathcal{X}) \text{ and } v_\chi(w_j) = 0 \text{ for } j = 1, 2, \dots, m.$$

The words w_j can be rewritten as words, say u_j , in the conjugates $a_{j,\ell}$; if this is done, only non-negative indices ℓ will occur (because of the hypothesis that $v_\chi(w_j) = 0$.) Let μ the largest value of ℓ that occurs in any of the words u_j and set

$$N_+ = \text{gp}(\{a_{j,\ell} \mid j = 1, \dots, m \text{ and } \ell = 0, \dots, \mu\}).$$

Then $t^{-1} \cdot N_+ \cdot t \subseteq N_+$. By applying the previous argument to $-\chi$, one finds similarly a positive integer ν for which the subgroup

$$N_- = \text{gp}(\{a_{j,\ell} \mid j = 1, \dots, m \text{ and } \ell = -\nu, \dots, -1, 0\}).$$

satisfies the condition $t \cdot N_- \cdot t^{-1} \subseteq N_-$. The subgroup $\text{gp}(N_+, t^\nu \cdot N_- \cdot t^{-\nu})$ is then invariant under conjugation by t , coincides therefore with N , and shows that N is generated by $m \cdot (1 + \max\{\mu, \nu\})$ elements.

A4.1b Comparison of the invariants of G and of G/N

If $\eta: \mathcal{X} \rightarrow G$ is a finite generating system of G , then $\pi \circ \eta: \mathcal{X} \rightarrow G \twoheadrightarrow Q$ is a finite generating system of Q . So the projection π induces a map $\pi_*: \Gamma(G, \mathcal{X}) \rightarrow \Gamma(Q, \mathcal{X})$ between the Cayley graphs; it is a covering map. This fact is the key ingredient in the proof of the following result which relates the invariants Σ^1 of G and Q :

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PROPOSITION A4.5 *Let $\bar{\chi}: Q \rightarrow \mathbb{R}$ a character of Q and let $\chi = \pi \circ \bar{\chi}: G \rightarrow \mathbb{R}$ denote its pull-back to G . Then the implication*

$$[\chi] \in \Sigma^1(G) \implies [\bar{\chi}] \in \Sigma^1(Q) \quad (\text{A4.2})$$

holds. Its converse is valid if N is a finitely generated group.

Proof. Since the $\bar{\chi}$ vanishes on N , the covering map $\pi_*: \Gamma(G, \mathcal{X}) \rightarrow \Gamma(Q, \mathcal{X})$ restricts to a surjective graph map $\pi_*: \Gamma_\chi \rightarrow \Gamma_{\bar{\chi}}$. If Γ_χ is connected, its quotient $\Gamma_{\bar{\chi}}$ is therefore also connected and so implication (A4.2) holds.

Conversely, assume $\Gamma_{\bar{\chi}}$ is connected. Given a vertex $g \in \Gamma_\chi$, there exists a path \bar{p} that runs in $\Gamma_{\bar{\chi}}$ and leads from 1_Q to $q = gN$. It can be lifted to a path p_1 that ends in g . This path runs in the subgraph Γ_χ and its origin g_1 is an element of N . Assume now N has a finite generating system \mathcal{A} , say. By the invariance property of Σ^1 (Theorem A2.3), one may arrange that \mathcal{A} is part of the generating system \mathcal{X} that is used in the construction of the Cayley graph Γ . Then there clearly exists a path p_0 from 1_G to g_1 that runs inside Γ_χ and thus the concatenated path $p_0.p_1$ connects 1_G to g in Γ_χ . \square

ADDENDUM A4.6 *The notation being as in Proposition A4.5, assume N is finitely generated and $Q = G/N$ abelian. Then $S(G, N) \subseteq \Sigma^1(G)$.*

Proof. Since Q is abelian, its invariant is all of $S(Q)$ by example A2.5 a); if, in addition, N is finitely generated Proposition A4.5 therefore implies that every non-zero character χ of G with $\chi(N) = \{0\}$ represents a point of $\Sigma^1(G)$. \square

A4.2 Proof of the Theorem A4.1

The proof of implication $S(G, N) \subseteq \Sigma^1(G) \Rightarrow N$ is finitely generated in Theorem A4.1 will be broken into several steps. We begin with two reductions and the introduction of coordinates on $S(G, N)$.

A4.2a Reduction to the case where Q is free abelian and choice of \mathcal{X}

Let $G, N \triangleleft G$ and $Q = G/N$ be as in in Theorem A4.1. Then Q is a finitely generated group abelian group; thus its torsion group $T(Q)$ is finite and the factor group $Q/T(Q)$ is free abelian. Let \hat{N} denote the preimage of $T(Q)$ under the canonical epimorphism $\pi: G \twoheadrightarrow Q$. Then N has finite index in \hat{N} and so N is finitely generated if, and only, \hat{N} has this property (see, e. g., [Rob96, p. 36, 1.6.18] for the non-trivial implication). Moreover, as the additive group of the field of reals \mathbb{R} is torsion-free every character that vanishes on N will vanish on \hat{N} ; so $S(G, N) = S(G, \hat{N})$.

It suffices therefore to prove the claim under the *additional hypothesis that Q be free abelian, say of rank k* . Accordingly, we choose the finite set of generators $\mathcal{X} = \mathcal{T} \cup \mathcal{Z}$ of G in such a way that $\pi: G \twoheadrightarrow Q$ maps \mathcal{T} bijectively onto a basis

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of Q and \mathcal{Z} is contained in N . In addition, we select \mathcal{Z} so that it contains all commutators $[y_1, y_2]$ with y_1, y_2 in $\mathcal{T} \cup \mathcal{T}^{-1}$.

A4.2b Filtering the Cayley graph

We next introduce coordinates on $S(G, N)$. Let $\vartheta: G \twoheadrightarrow Q = G/N \xrightarrow{\sim} \mathbb{Z}^k$ be an epimorphism of groups that sends \mathcal{T} onto the set of standard basis vectors of \mathbb{Z}^k and define \mathbb{Z}^k to be the standard lattice of the real vector space \mathbb{R}^k equipped with the usual inner product $\langle -, - \rangle$ and norm $\| - \|$. Similarly as in section A1.1d, we introduce then coordinates on the sphere $S(G, N)$ by assigning to $u \in \mathbb{S}^{k-1}$ the character $\chi_u: G \rightarrow \mathbb{R}$ given by $\chi_u(g) = \langle u, \vartheta(g) \rangle$.

With the help of these coordinates, we construct a filtration on G by *spherical* subsets $G(\rho)$. These subsets are defined thus:

$$G(\rho) = \{g \in G \mid \|\vartheta(g)\| \leq \rho\}. \quad (\text{A4.3})$$

Here ρ ranges over the discrete subset $\{\sqrt{m} \mid m \in \mathbb{N}\}$ of \mathbb{R} . Since N is the kernel of ϑ , each $G(\rho)$ is N -invariant; but it is neither a subgroup nor a submonoid of G .

Let $\Gamma(\rho) \subseteq \Gamma = \Gamma(G, \mathcal{X})$ to be the full subgraph with vertex set $G(\rho)$. Then N acts on $\Gamma(\rho)$ and this action is free, for it is induced by the action of N on the Cayley graph Γ . The quotient graph $N \backslash \Gamma(\rho)$ is finite. If it is in connected for some sufficiently large radius ρ then N will act freely on the connected graph $\Delta = \Gamma(\rho)$ with finite quotient graph $N \backslash \Delta$ and will therefore be finitely generated by Proposition A4.7 below.

A4.2c Search for a connected subgraph $\Gamma(\rho_0)$

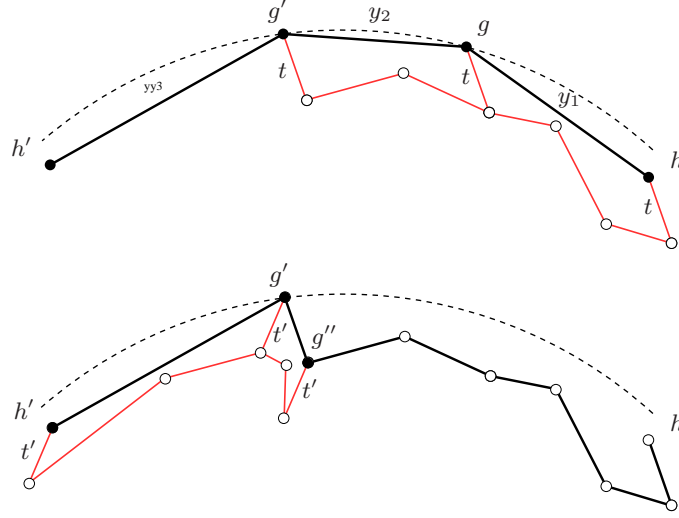
We next aim at proving that the graph $\Gamma(\rho)$ is connected whenever the radius ρ is large enough. The intuitive reason is this. Let ρ_0 be a large radius and let g_0 be a vertex in $\Gamma(\rho_0)$. There exists then a path p in the Cayley graph $\Gamma = \Gamma(G, \mathcal{X})$ from 1 to g_0 . If p runs inside $\Gamma(\rho_0)$ all is well; otherwise, let $\rho_p \in \mathbb{R}$ be the smallest real number ρ for which p is contained in the subgraph $\Gamma(\rho)$, and let M_p be the set of all vertices g of p with $\|\vartheta(g)\| = \rho_p$. Consider now a maximal consecutive sequence of vertices (g, g', \dots) in M_p . Let (g, y_1) be the *inverse* of the edge of p that *terminates* in g and let (g, y_2) be the edge of p that *begins* in g . Define u to be the vector $u = -\vartheta(g)/\|\vartheta(g)\|$. Choose an element $t \in \mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ with $\chi_u(t) > 0$. By Theorem A3.1 there is then a path p_{y_1} from t to $y_1 t$, and a path p_{y_2} from t to $y_2 t$, all in such a way that $v_u(p_{y_1}) > v_u(1, y_1)$ and $v_u(p_{y_2}) > v_u(1, y_2)$; here v_u is short for v_{χ_u} . The norm of $\vartheta(g y_1)$ does not exceed that of $\vartheta(g y_2)$ and so $\chi_u(y_1) \geq 0$. Similarly, $\chi_u(y_2) \geq 0$. The paths p_{y_1} and p_{y_2} satisfy therefore the inequalities

$$v_{\chi_u}(p_y) > 0 \text{ for } y = y_1 \text{ and } y = y_2. \quad (\text{A4.4})$$

Replace now the subpath $(h, y_1^{-1} y_2)$ of p by the path

$$q = (h, t) \cdot (g \cdot p_{y_1})^{-1} \cdot (g \cdot p_{y_2}) \cdot (g', t)^{-1},$$

Notes

Figure A.9: One step in the transformation of the path p

ending up with a path p' that contains no vertex z between $h = gy_1$ and $g' = gy_2$ with $\|\vartheta(z)\| = \rho_p$. The last of the edges of the subpath q is $(g', t)^{-1}$. Now apply to the inverse of this edge and the edge (g', y_3) the analogous transformation. Continuing in this way one will finally arrive at a path from 1 to g_0 all whose vertices z satisfy the inequality z with $\|\vartheta z\| < \rho_p$. Because the norm of a lattice point $\vartheta(z)$ lies in the discrete set $\{\sqrt{m} \mid m \in \mathbb{N}\}$, iteration of the described transformation of paths will lead in finitely many steps to a path from 1 to the given g_0 that runs in $\Gamma(\rho_0)$.

The crux of the proof thus lies in finding a radius ρ_0 that allows one to carry out the described transformation of paths. To find it, we bring into play the hypothesis that the subsphere $S(G, N)$ of $S(G)$ is contained in $\Sigma^1(G)$. We can construct an open cover of the sphere $\mathbb{S}^{k-1} \xrightarrow{\sim} S(G, N)$ as follows: given a unit vector $u \in \mathbb{S}^{k-1}$, choose an element $t \in \mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ with $\chi_u(t) > 0$. Then find, with the help of Theorem A3.1, for each $y \in \mathcal{Y}$, a path $p_y(u)$ that leads from t to yt and satisfies the inequality $v_u(p_y(u)) > v_u((1, y))$. The couple $\psi_u = (t, \{p_y(u)\})$ gives rise to a real valued function f_{ψ_u} ; it is defined on \mathbb{S}^{k-1} and given by

$$f_{\psi_u}(u') = \min\{v_{u'}(p_y(u)) - v_{u'}((1, y)) \mid y \in \mathcal{Y}\}. \quad (\text{A4.5})$$

This function enters into the definition of a neighbourhood of u , namely

$$\mathcal{N}_u = \mathcal{N}_u(\psi_u) = \{u' \in \mathbb{S}^{k-1} \mid f_{\psi_u}(u') > 0\}. \quad (\text{A4.6})$$

The set \mathcal{N}_u is an open subset of \mathbb{S}^{k-1} . As \mathbb{S}^{k-1} is compact, there exist a finite family $\mathcal{F} = \{u_1, \dots, u_\ell\}$ of unit vectors such that the collection of open subsets $\{\mathcal{N}_{u_j} \mid 1 \leq j \leq \ell\}$ covers \mathbb{S}^{k-1} . This family is next used in the definition of an

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auxiliary function $\varepsilon: \mathbb{S}^{k-1} \rightarrow \mathbb{R}$; it is given by

$$\varepsilon(u') = \max\{f_{\psi_j}(u') \mid 1 \leq j \leq \ell\}. \quad (\text{A4.7})$$

Here ψ_j is short for ψ_{u_j} . The definition of ε and of the neighbourhoods $\mathcal{N}_{u_j}(\psi_j)$ imply that the function ε is positive on the sphere \mathbb{S}^{k-1} . As it is continuous and \mathbb{S}^{k-1} is compact, it admits therefore a positive lower bound, say ε_0 .

To define the radius ρ_0 we need a last definition. For each index $j \in \{1, \dots, \ell\}$ and each generator $y \in \mathcal{Y}$, let $\mathcal{S}_{j,y}$ be the set of all vertices of the path $p_y(u_j)$. Define then an auxiliary radius r by setting

$$r = \max\{\|\vartheta(h)\| \mid h \in \mathcal{S}_{j,y} \text{ and } (j, y) \in \{1, \dots, \ell\} \times \mathcal{Y}\}. \quad (\text{A4.8})$$

Set now $\rho_0 = \frac{1}{2}r^2/\varepsilon_0$.

Consider a vertex g on a path from 1 to a given vertex $g_0 \in \Gamma(\rho_0)$ for which $\|\theta(g)\|$ is maximal, and put $z = \theta(g)$, $\rho = \|z\|$ and $u = -\theta(g)/\rho$. It $\rho \leq \rho_0$, all is well; otherwise, choose an index j with $f_{\psi_j}(u) \geq \varepsilon_0$. Let $y_0 \in \mathcal{Y}$ be a generator with $\chi_u(y_0) \geq 0$. The starting point of the path $p_{y_0}(u_j)$ is t and $v_u(p_{y_0}(u_j)) > v_u((1, y_0)) = 0$, and so $\chi_u(t) > 0$. Define (g, y_1) to be the inverse of the edge of p that terminates in g and (g, y_2) to be the edge of p that begins in g . Then formula (A4.4) applies to the generators y_1 and y_2 . Given a group element $h \in \mathcal{S}_{j,y_1} \cup \mathcal{S}_{j,y_2}$, set $y = \vartheta(h)$. Then

$$\begin{aligned} \|z + y\|^2 &= \|z\|^2 + 2\langle z, y \rangle + \|y\|^2 = \rho^2 + \|y\|^2 - 2\langle u \cdot \rho, y \rangle \\ &\leq \rho^2 + r^2 - 2\varepsilon_0 \cdot \rho < \rho^2. \end{aligned}$$

We conclude that the family of couples $\{\psi_{u_j} = (t_{u_j}, \{p_y(u_j)\}_{y \in \mathcal{Y}}) \mid 1 \leq j \leq \ell\}$ permits one to transform every the path p from 1 to a vertex g_0 in the graph $\Gamma(\rho_0)$ into a path that runs inside this graph. The subgraph $\Gamma(\rho_0)$ is thus connected.

The proof of Theorem A4.1 is now complete save for the proof of Proposition A4.7 below.

A4.2d A geometric criterion for finite generation

The following criterion for the finite generation of a group acting on a connected graph is well-known.

PROPOSITION A4.7 *Let G be a group acting on a connected graph Γ so that the following two conditions are satisfied:*

- (i) *the factor graph $G \backslash \Gamma$ is finite;*
- (ii) *the stabilizers $G_v = \{g \in G \mid gv = v\}$ of all vertices $v \in \Gamma$ are finitely generated.*

Then G is finitely generated.

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Proof. Let $\pi: \Gamma \twoheadrightarrow G \backslash \Gamma$ be the canonical projection. Choose a finite subgraph $\Gamma_0 \subseteq \Gamma$ with $\pi(\Gamma_0) = G \backslash \Gamma$. Next fix, for each couple (v, v') of distinct vertices of Γ_0 lying in the same G -orbit, an element $t \in G$ with $v = t.v'$ and let \mathcal{T} be the (finite) set of these elements $t \in G$. Define K to be the subgroup of G generated by \mathcal{T} and by the stabilizers G_v of the vertices of Γ_0 ; then K is a finitely generated.

The group K admits of an alternative description, namely

$$K = \text{gp}(\{g \in G \mid g.\Gamma_0 \cap \Gamma_0 \neq \emptyset\}). \quad (\text{A4.9})$$

Indeed, if $g.\Gamma_0 \cap \Gamma_0 \neq \emptyset$ then Γ_0 contains vertices v and w with $g.v = w$. If $v = w$ then $g \in G_v$; otherwise there is some $t \in \mathcal{T}$ with $t.v = w$ and so g is a product $t \cdot g'$ with $g' \in G_v$. So the right hand side of equation (A4.9) is contained in K . The opposite inclusion is obvious.

We assert that G equals K . To see this, consider an element of $g \in G$. Pick $v_0 \in V(\Gamma_0)$. Since Γ is connected, there exists a path from v_0 to $g.v_0$ say

$$p = (v_0 = w_0, w_1, \dots, w_\ell = g.v_0).$$

Since $V(\Gamma) = G.V(\Gamma_0)$ there exists for every index $j \in \{1, \dots, \ell - 1\}$ a vertex $v_j \in \Gamma_0$ and an element $g_j \in G$ with $w_j = g_j.v_j$. Moreover, $w_0 = 1.v_0$ and $w_\ell = g.v_\ell$. Similarly, for $j \in \{1, \dots, \ell\}$, there exist for the edge f_j from w_{j-1} to w_j an edge $e_j \in E(\Gamma_0)$ and a group element $h_j \in G$ with $f_j = h_j.e_j$. One has $w_0 = 1.v_0$ and $1 \in K$. Assume, inductively, that $g_{\ell-1} \in K$. Since $h_\ell.\Gamma_0$ intersects $g_{\ell-1}.\Gamma_0$ the intersection $(g_{\ell-1}^{-1}h_\ell).\Gamma_0 \cap \Gamma_0$ is non-empty whence $g_{\ell-1}^{-1}h_\ell \in K$ and so $h_\ell = g_{\ell-1} \cdot k'$ for some $k' \in K$. Similarly one sees that $g_\ell = h_\ell \cdot k''$ for some $k'' \in K$. So $g_\ell = h_\ell \cdot k'' = g_{\ell-1} \cdot k' \cdot k''$; as $g_{\ell-1} \in K$ by the induction hypothesis this shows that $g \in K$. \square

A4.3 Application: Normal subgroups of large co-rank

We conclude section A4 with some applications of Theorem A4.1.

A4.3a Finding finitely generated normal subgroups of large co-rank

Our applications of Theorem A4.1 will actually be corollaries of Theorem A4.8 given below. In it, the *complement* of the invariant is assumed to be contained in a finite union of subspheres and one aims at finding a finitely generated normal subgroup N containing G' with G/N having large torsion-free rank.

THEOREM A4.8 *Assume G is a finitely generated group for which the complement of the invariant satisfies an inclusion of the form*

$$\Sigma^1(G)^c \subseteq \bigcup \{S(G, H_j) \mid j \in J\}, \quad (\text{A4.10})$$

each H_j being a subgroup of G and J being finite. Then the following claims hold:

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(i) there exists a finitely generated normal subgroup $N \supseteq G'$ with

$$r_0(G/N) = \min\{r_0((H_j G')/G') \mid j \in J\}. \quad (\text{A4.11})$$

(ii) if equality holds in (A4.10) then no normal subgroup $N \supseteq G'$ with

$$r_0(G/N) > \min\{r_0((H_j G')/G') \mid j \in J\} \quad (\text{A4.12})$$

can be finitely generated.

Proof. Let N be a normal subgroup containing the derived group G' . According to Theorem A4.1, N is finitely generated if, and only if, the subsphere $S(G, N)$ and the complement $\Sigma^1(G)^c$ of the invariant are disjoint. If this complement satisfies inclusion (A4.10), N will therefore be finitely generated if the intersections $S(G, N) \cap S(G, H_j)$ are empty for every index $j \in J$. Now $S(G, N) \cap S(G, H_j)$ is empty if, and only if, every character $\chi: G \rightarrow \mathbb{R}$ which vanishes on both N and H_j is zero or, put differently, if the group $N \cdot H_j$ has finite index in G .

Let $\vartheta: G \twoheadrightarrow G/G' \twoheadrightarrow \mathbb{Z}^n$ be an epimorphism of G onto the standard lattice of rank $n = r_0(G_{\text{ab}})$ and let $L = \vartheta(N)$ and $L_j = \vartheta(H_j)$ be the images of N and H_j under ϑ . Then $L + L_j$ has finite index in \mathbb{Z}^n and so the subspaces U, U_j of \mathbb{Q}^n spanned by L and L_j , respectively, satisfy the relation $U + U_j = \mathbb{Q}^n$.

Conversely, if U and U_j are subspaces of \mathbb{Q}^n satisfying $U + U_j = \mathbb{Q}^n$ then the subgroups $N = \vartheta^{-1}(U \cap \mathbb{Z}^n)$ and $H_j = \vartheta^{-1}(U_j \cap \mathbb{Z}^n)$ are normal subgroups and the subspheres $S(G, N)$ and $S(G, H_j)$ are disjoint.

The preceding reasoning shows that Theorem A4.8 can be reduced to the following claim about finite dimensional vector spaces:

LEMMA A4.9 *Given a finite-dimensional vector space V over an infinite field \mathbb{F} and subspaces U_1, \dots, U_ℓ , there exists a subspace U of co-dimension*

$$m = \min\{\dim U_j \mid 1 \leq j \leq \ell\}$$

which is a simultaneous supplement to all the subspaces U_j . Moreover, there does not exist a simultaneous supplement of co-dimension strictly larger than m .

The proof of Lemma A4.9 depends on another, more widely known, result about vector spaces, namely

LEMMA A4.10 *A finite-dimensional vector space V over an infinite field cannot be covered by finitely many proper subspaces.*

Lemma A4.9 will be proved by induction on $d = \dim V - m$. The claim is patent for $d = 0$. Assume now $d \geq 1$ and set $J_1 = \{j \in J \mid U_j \neq V\}$. By Lemma A4.10 there exists a vector v_1 which lies outside every U_j with $j \in J_1$. Set $\overline{V} = V/(F \cdot v_1)$ and let $\pi: V \twoheadrightarrow \overline{V}$ denote the canonical projection. For every

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$j \in J_1$, the composition of $U_j \hookrightarrow V$ with π is injective, and so the subspaces $\overline{U_j} = \pi(U_j)$ of \overline{V} form a finite family of subspaces with

$$\dim(\overline{V}) - \min\{\dim \overline{U_j} \mid j \in J_1\} = d - 1.$$

By the inductive hypothesis, there exists therefore a supplement $\overline{U} \subset \overline{V}$ of dimension $d - 1$. Its preimage $U = \pi^{-1}(\overline{U})$ is a supplement of each proper subspace U_j with $j \in J_1$, and hence of every subspace U_j with $j \in J$.

The addendum is clear and so we are left with establishing Lemma A4.10; this will be by induction on $d = \dim V$. If $d = 1$, the claim is evident, if $d = 2$ it holds since V contains infinitely many lines passing through the origin. Assume now that $d \geq 3$ and let U_1, \dots, U_ℓ be proper subspaces of V . Some of them may have co-dimension 1, but as V contains infinitely many subspaces of co-dimension 1, there exist a subspace U of co-dimension 1 with $U \neq U_j$ for all indices j , whence each of the subspaces $U'_j = U_j \cap U$ is a proper subspace of U . As $\dim U = d - 1$, the inductive hypothesis applies and provides one with a vector $v \in U$, that lies in none of the intersections $U'_j = U_j \cap U$. But, if so, v does not lie in one of the original subspaces U_j , either. \square

REMARK A4.11 Lemma A4.9 may be well-known. The given proof is a variant of an argument found in April of 2008 by Simon Kurmann and Fred Rohrer, two students of Markus Brodmann (University of Zürich).

EXAMPLE A4.12 Assume G is the internal direct product of subgroups G_1, G_2 and let $\pi_i: G \rightarrow G_i$ be the canonical projections onto the factors G_i . Proposition A2.7 expresses $\Sigma^1(G)^c$ in terms of the subsets $\Sigma^1(G_i)^c$ and implies the upper bound

$$\Sigma^1(G)^c = \pi_1^*(\Sigma^1(G_1))^c \cup \pi_2^*(\Sigma^1(G_2))^c \subseteq S(G, G_2) \cup S(G, G_1). \quad (\text{A4.13})$$

Theorem A4.8 then allows us to find a normal subgroup N containing $G' = G'_1 \cdot G'_2$ and so that

$$\begin{aligned} r_0(G/N) &= \min\{r_0((G_2 \cdot G')/G'), r_0((G_1 \cdot G')/G')\} \\ &= \min\{r_0((G_1)_{\text{ab}}), r_0((G_2)_{\text{ab}})\}. \end{aligned} \quad (\text{A4.14})$$

In this particular example, claim (i) of Theorem A4.8 can be established by a simple, direct argument. For $i \in \{1, 2\}$, let M_i denote the preimage of the torsion-subgroup of $(G_i)_{\text{ab}}$ under the canonical map $G_i \twoheadrightarrow (G_i)_{\text{ab}}$. The quotient groups $A_1 = G_1/M_1$ and $A_2 = G_2/M_2$ are free-abelian of rank n and n' , say. Let $\{x_1, x_2, \dots, x_n\}$ be a subset of G_1 that projects onto a basis of A_1 ; similarly, let $\{x'_1, x'_2, \dots, x'_{n'}\}$ be a subset of G_2 projecting onto a basis of A_2 . Since A_1 is a finitely presentable group, there exist a finite subset $\mathcal{M}_1 \subset M_1$ which generates M_1 over the group $\text{gp}(\{x_1, x_2, \dots, x_n\})$ (see, e. g., [Rob96, p. 53, 2.2.3]). Similarly, M_2 is generated over $\text{gp}(\{x'_1, \dots, x'_{n'}\})$ by a finite subset $\mathcal{M}_2 \subset M_2$.

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We are now ready to construct a finitely generated normal subgroup N of G which contains G' for which G/N has torsion-free rank $m = \min\{n, n'\}$. Assuming, as we may, that $n \leq n'$ we define N to be the subgroup generated by the finite set

$$\mathcal{M}_1 \cup \mathcal{M}_2 \cup \{x_i \cdot x'_i \mid 1 \leq i \leq n\} \cup \{x'_i \mid n < i \leq n'\}.$$

Since x_i and $x_i \cdot x'_i$ induce by conjugation the same actions on M_1 , the normal subgroup M_1 is contained in N ; similarly, one sees that N contains M_2 . Finally, G/N is free abelian of rank $m = \min\{r_0((G_1)_{\text{ab}}), r_0((G_2)_{\text{ab}})\}$.

REMARK A4.13 There exist several classes of groups G for which $\Sigma^1(G)^c$ is a finite union of subspheres of the form $S(G, H)$. Graph groups provide such a class; it will be treated below. Another class consists of groups of automorphisms of free groups studied by J. McCool in [McC86]; the invariants of these groups have been determined by L. A. Orlandi-Korner in [OK00]. A third class is formed by the fundamental groups of compact Kähler manifolds (see [Del10]).

A4.3b Finitely generated normal subgroups in graph groups

Right angled Artin groups, or *graph groups* as they used to be known, are given by a presentation whose relations are commutator relations among the generators. Such a presentation can be described by a finite combinatorial graph $\Delta = (V(\Delta), E(\Delta))$. Its set of vertices $V(\Delta)$ is the set of distinguished generators $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ of the group. Each edge $\{x_i, x_j\}$ of Δ corresponds to the relation $x_i x_j = x_j x_i$, and there are no further defining relations. The graph group G_Δ is thus given by

$$G_\Delta = \langle x_1, x_2, \dots, x_n \mid x_i x_j = x_j x_i \text{ for every edge } \{x_i, x_j\} \in E(\Delta) \rangle. \quad (\text{A4.15})$$

Their simple presentations notwithstanding, graph groups are surprisingly varied, a fact that has led to some remarkable applications (cf. Ruth Charney's survey [Cha07]).

The invariant of a graph group can be described in terms of subsets $\mathcal{S} \subseteq \mathcal{X}$ whose removal results in a disconnected graph; such subsets are called *separating* (cf. [Die10, p. 11]). One has

PROPOSITION A4.14 *Let $G = G_\Delta$ be a graph group with graph Δ . If Δ is a complete graph, G is free-abelian of rank $\text{card } V(\Delta)$ and $\Sigma^1(G) = S(G)$. Otherwise, the complement of its invariant is given by*

$$\Sigma^1(G)^c = \bigcup_{\mathcal{S}} S(G, \text{gp}(\mathcal{S})) \quad (\text{A4.16})$$

where \mathcal{S} runs over the minimal separating subsets of $V(\Delta)$.

Proposition A4.14 will be established in section B1.3. Upon allying it with Theorem A4.8 one arrives at a result going back to J. Meier and L. VanWyck (see [MV95, Theorem 6.3]):

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COROLLARY A4.15 Assume Δ is not the complete graph on $V(\Delta)$ and G_Δ is graph group with graph Δ . Then the number

$$\max\{r_0(G/N) \mid N \text{ is a finitely generated group containing } G'_\Delta\}$$

coincides with the cardinality of a separating subset of $V(\Delta)$ having least number of vertices.

REMARK A4.16 A finitely generated normal subgroup $N \triangleleft G_\Delta$ enjoying the property stated in Corollary A4.15 can be found as follows. Let $\{x_1, x_2, \dots, x_n\}$ be the vertex set of Δ and let $\vartheta: G_\Delta \rightarrow \mathbb{Z}^n$ be the epimorphism sending x_i to the i -th standard basis element of \mathbb{Z}^n . Let m be the cardinality of a minimal separating set of Δ . Formula (A4.16) and the proof of Theorem A4.8 then show that it suffices to find a free abelian subgroup A of rank $n - m$ in \mathbb{Z}^n that is transversal to every subgroup $B \subset \mathbb{Z}^n$ generated by m basis vectors, and then to set $N = \vartheta^{-1}(A)$. An explicit example of A is the subgroup generated by the $n - m$ rows $(1^k, 2^k, \dots, n^k)$ where k ranges from 0 to $n - m - 1$ (use the Vandermonde determinant).

EXAMPLES A4.17 a) Assume Δ has $n \geq 2$ vertices and is not a complete graph. If Δ is not connected, the empty set is separating and Corollary A4.14 implies that $\Sigma^1(G_\Delta)$ is empty. Theorem A4.1 then shows that G_Δ has no finitely generated normal subgroup N whose factor group G_Δ/N is an infinite abelian group. If, on the other hand, Δ is connected every separating set has at least one vertex. By Corollary A4.15 the group G_Δ admits therefore finitely generated normal subgroups N with infinite cyclic quotient G_Δ/N . This conclusion is well-known; indeed, by [MV95, Thm. 6.1], the kernel of every rank 1 character $\chi: G \rightarrow \mathbb{Z}$ is finitely generated if χ sends every generator $x_i \in V(\Delta)$ to a non-zero integer.

b) Let Δ be a tree with $n \geq 3$ vertices. Every vertex x_j of degree greater than 1 separates the tree, while the removal of a vertex of degree 1 results in a connected subtree. So the minimal separating subsets S are the singletons constituted by the vertices of degree greater than 1. Each such singleton gives rise to a subsphere $S(G, S)$ of co-dimension 1.

c) Let Δ_n be an n -gon with $n \geq 4$ sides. No singleton and no pair of adjacent vertices separates the polygon, but every pair of non-adjacent vertices does. By Proposition A4.14 the complement of the invariant is therefore the union

$$\bigcup \{S(G, \text{gp}(\{x_i, x_j\})) \mid j \notin \{i-1, i, i+1\}\}. \quad (\text{A4.17})$$

of subspheres of co-dimension 2. According to Corollary A4.15, the group G contains a finitely generated normal subgroup N with $G/N \approx \mathbb{Z}^2$, but no finitely generated normal subgroup N with $G/N \approx \mathbb{Z}^3$.

d) The reader can find further examples in section B1.3b.

A5 Finitely related groups and Σ^1

In this section, a second main result about the invariant Σ^1 will be established. It provides a necessary condition that every finitely related soluble group G (with G_{ab} of positive rank) must satisfy; it is this necessary condition that prompted R. Bieri and R. Strebel in the late 1970s to introduce $\Sigma^0(G; A)$, a precursor of the protagonist of this chapter.

The result has two versions. The local one, Theorem A5.7, deals with the kernel N of a non-zero character χ which is such that both $[\chi]$ and $-[\chi]$ lie outside $\Sigma^1(G)$. It assumes that G admits a finite presentation and asserts that N has a decomposition $S_- \star_{S_0} S_+$ as a free product with amalgamation, in which the amalgam S_0 has infinite index in both factors. This implies that N contains non-abelian free subgroups. The global version arises from the first one by assuming that G has no non-abelian free subgroups. Then $\Sigma^1(G)$ must contain at least one point out of each pair $\{[\chi], -[\chi]\}$ of antipodal points. If one denotes the set of all antipodes of points of a subset $\Sigma \subseteq S(G)$ by $-\Sigma$, this result can be stated as

THEOREM A5.1 *Let G be a finitely presented group that contains no non-abelian free subgroup. Then*

$$\Sigma^1(G) \cup -\Sigma^1(G) = S(G). \quad (\text{A5.1})$$

In particular, $\Sigma^1(G)$ is non-empty unless G/G' is finite.

By combining this result with corollary A4.3 one obtains the rather surprising

COROLLARY A5.2 *Let G be a finitely presented group that does not contain a non-abelian free subgroup. If $r_0(G_{\text{ab}}) \geq 2$ then G contains finitely generated normal subgroups $N \triangleleft G$ with infinite cyclic factor group G/N .*

Proof. Since the torsion-free rank of G_{ab} is greater than 1, the sphere $S(G)$ is connected; so it can not be written as the disjoint union of two non-empty open subsets. Theorem A5.1, on the other hand, shows that $S(G)$ is the union of two open subsets, $\Sigma^1(G)$ and $-\Sigma^1(G)$. The intersection $\Sigma^1(G) \cap -\Sigma^1(G)$ is therefore a non empty open set and so it contains a rank 1 point, say $[\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}]$ (by Lemma B3.24 rank 1 points are dense in $S(G)$). Corollary A4.3 then implies that the kernel of χ is a finitely generated group. \square

REMARKS A5.3 a) The conclusion of Theorem A5.1 need not hold if G does not admit a finite presentation or if it contains a non-abelian free subgroup. The first claim can be substantiated by the metabelian groups discussed in example A3.6: if the fraction $p/q \in \mathbb{Q}$ occurring in this example is in reduced form and if neither p nor q is a unit in the ring \mathbb{Z} , then $\Sigma^1(G)$ is empty. The second claim is justified by the non-abelian free groups of finite rank: these groups are finitely related, their invariants, however, are empty (see example 3 in section A2.1a).

Notes

- b) The conclusion of Corollary A5.2 need not be true if G is a finitely related soluble group whose abelianization has rank 1. Witnesses are the metabelian Baumslag-Solitar groups $\langle a, t \mid tat^{-1} = a^m \rangle$ for $m > 1$.
- c) Theorem A5.1 goes back to Theorem C in [BNS87].

A5.1 Main result for finitely related groups

Throughout this section, G will denote a finitely generated group, with finite generating system $\eta: \mathcal{X} \rightarrow G$, and $\chi: G \rightarrow \mathbb{R}$ a non-zero character. Our first aim is to construct a collection of connected graphs $\Delta_b = \Delta_b(\chi)$ that come equipped with a canonical action of $N = \ker \chi$. We shall prove that the quotient graphs $N \backslash \Delta_b$ are isomorphic to segments, if the real number b satisfies condition (A5.2), and that they are trees whenever G admits a finite presentation, and b is sufficiently large (and satisfies the technical condition (A5.2)).

A5.1a Definition and analysis of the graphs Δ_b

Given a real number $b \geq 0$, let D_+ denote the set of the components of the subgraph $\Gamma_+ = \Gamma_\chi$, let D_- be the set of components of $\Gamma_- = \Gamma_\chi^{(-\infty, b]}$ and D_0 the set of components of the slice $\Gamma_0 = \Gamma_\chi^{[0, b]}$. The kernel $N = \ker \chi$ acts on the three subgraphs Γ_+ , Γ_- and Γ_0 and hence on the three sets D_+ , D_- and D_0 . These sets give rise to a bipartite, oriented graph $\Delta = \Delta_b(\chi)$ that is equipped with an N -action:

DEFINITION A5.4 The oriented graph $\Delta_b(\chi)$ has vertex set $V(\Delta_b) = D_- \cup D_+$ and edge set $E(\Delta_b) = D_0$. The origin and terminus of an edge are given by

$$\begin{aligned} \iota(\mathcal{E}) &= \text{path component of } \Gamma_- = \Gamma_\chi^{(-\infty, b]} \text{ containing } \mathcal{E}, \\ \tau(\mathcal{E}) &= \text{path component of } \Gamma_+ = \Gamma_\chi \text{ containing } \mathcal{E}. \end{aligned}$$

The next result summarizes some key properties of the graphs $\Delta_b(\chi)$.

PROPOSITION A5.5 Let \mathcal{E}_0 be the component of $\Gamma_0 = \Gamma_\chi^{[0, b]}$ that contains the vertex 1 and set $\mathcal{C}_- = \iota(\mathcal{E}_0)$ and $\mathcal{C}_+ = \tau(\mathcal{E}_0)$. Let S_0 , S_- and S_+ be the stabilizers of the edge \mathcal{E}_0 and of its end points \mathcal{C}_- , \mathcal{C}_+ , and assume the real number b satisfies

$$b \geq 2 \cdot \max\{|\chi(x)| \mid x \in \mathcal{X}\} \text{ and } b \in \text{im } \chi. \quad (\text{A5.2})$$

Then the graph $\Delta_b = \Delta_b(\chi)$ and the action of N on it have the following properties:

- (i) Δ_b is connected;
- (ii) the quotient graph $N \backslash \Delta_b$ is an edge with distinct end points;
- (iii) if S_0 has finite index in S_+ or if S_+ is finitely generated then $-\chi \in \Sigma^1(G)$;
- (iv) if S_0 has finite index in S_- or if S_- is finitely generated then $\chi \in \Sigma^1(G)$.

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(v) Assume, in addition, that the group G is finitely related. Then Δ_b is a tree whenever b is sufficiently large.

Proof. We first describe a procedure that associates to every path p in the Cayley graph $\Gamma = \Gamma(G, \mathcal{X})$ a unique path $\sigma(p)$ in the graph $\Delta_b(\chi)$. The path p is a sequence $(g_1, g_2, \dots, g_\ell)$ of adjacent vertices. Set

$$I = \{j \in \{1, 2, \dots, \ell\} \mid \chi(g_j) \notin [0, b]\}.$$

This set may be empty. If so, the path runs in a single component \mathcal{E} of the slice $\Gamma_0 = \Gamma_\chi^{[0, b]}$ and we define $\sigma(p)$ to be the path of length 0 which starts at, and ends in, the terminus of the edge \mathcal{E} . Otherwise, I is a union of subintervals I_1, I_2, \dots, I_f made up of consecutive integers. For each subinterval $I_k = (j_k, j_k + 1, \dots, j_k + m_k)$, the subpath $(g_{j_k}, \dots, g_{j_k + m_k})$ runs in a single component $v \in D_- \cup D_+$. Denote this component by v'_k . It may happen that two consecutive vertices, say v'_j and v'_{j+1} , coincide; if so, replace each maximal constant subsequence of v'_1, v'_2, \dots, v'_f by its first member; let $\sigma(p) = (v_1, v_2, \dots, v_h)$ be the sequence so obtained. Then $\sigma(p)$ is a sequence of adjacent vertices of $\Delta_b(\chi)$ and thus a path in this graph.

(i) Given vertices v and v' of $V(\Delta_b) = D_+ \cup D_-$, we have to find an edge path in Δ_b from v to v' . If $v = v'$, the empty path will do; so assume $v \neq v'$. The vertex v is a component of either Γ_+ or Γ_- ; let $g \in v$ be a group element that is in this component, but outside the slice $\Gamma_{0, b} = \Gamma_\chi^{[0, b]}$; similarly, let $g' \in v'$ be a vertex outside this slice. Since the Cayley graph is connected, there exists a path p from g to g' . This path defines a sequence $\sigma(p)$ in $\Delta_b(\chi)$ that leads from v to v' .

(ii) By Lemma A3.7 the group $N = \ker \chi$ acts transitively, both on the set of components D_+ of Γ_χ and on the set of components D_- of $\Gamma_\chi^{(-\infty, b]}$. As N maps every component of D_+ to a component of D_+ and every component of D_- to a component of D_- , the quotient graph $N \backslash \Delta_b$ has exactly two vertices.

Consider now an edge \mathcal{E}_1 of the graph $\Delta_b(\chi)$ that terminates in the vertex \mathcal{C}_+ . Then \mathcal{E}_1 is a component of the intersection $\Gamma_\chi^{[0, b]} \cap \mathcal{C}_+$; by Lemma A5.6 below it contains therefore a vertex $h_1 \in S_+$. Its inverse h_1 then maps \mathcal{E} onto \mathcal{E}_0 . All taken together, we have shown that the quotient graph $N \backslash \Delta_b$ is a single edge with distinct end points.

(iii) According to lemma A5.6 every component \mathcal{E} of $\Gamma_\chi^{[0, b]} \cap \mathcal{C}_+$ contains a vertex in S_+ ; so the action of S_+ on these components is transitive. If S_0 has finite index in S_+ the intersection $\Gamma_\chi^{[0, b]} \cap \mathcal{C}_+$ has therefore finitely many components, say $\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_k$. For each $j \in \{1, 2, \dots, k\}$ one can find a path p_j in Γ_χ that links the vertex $1 \in \mathcal{E}_0$ to a vertex of \mathcal{E}_j . Let $b' > b$ be a real number that satisfies condition (A5.2) and is so large that all these paths run inside the slice $\Gamma_\chi^{[0, b']}$. The components $\mathcal{E}_0, \dots, \mathcal{E}_k$ lie then in a single component of $\Gamma_\chi^{[0, b']}$. The intersection $\Gamma_\chi^{[0, b']} \cap \mathcal{C}_+$ is therefore connected; indeed, every of its components contains a vertex in $\Gamma_\chi^{[0, b]}$ and hence a vertex of one of the subgraphs $\mathcal{E}_1, \dots, \mathcal{E}_k$. Theorem A3.8 then allows us to conclude that $-\chi$ is in $\Sigma^1(G)$. If S_+ is finitely generated, a similar

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argument shows that the intersection $\Gamma_\chi^{[0,b']} \cap \mathcal{C}_+$ is connected for some $b' > b$, and so it follows as before that $-\chi$ is in $\Sigma^1(G)$.

(iv) By the choice of b , there exists an element $g \in G$ with $\chi(g) = b$. Its inverse g^{-1} induces an isomorphism of graphs $\Gamma(G, \mathcal{X}) \xrightarrow{\sim} \Gamma(G, \mathcal{X})$ that maps the subgraph $\Gamma_- = \Gamma_\chi^{(-\infty, b]}$ onto the graph $\Gamma_\chi^{(-\infty, 0]} = \Gamma_{-\chi}$. As $\ker(-\chi) = \ker \chi$, the claim about Γ_- reduces therefore to that about $\Gamma_{-\chi}$.

(v) We assume now that G admits a finite presentation, say $\pi: \langle \mathcal{X}; \mathcal{R} \rangle \xrightarrow{\sim} G$. Each relator $r \in \mathcal{R}$ defines a closed path $p_{1,r} = (1, r)$ in the Cayley graph $\Gamma(G, \mathcal{X})$. Let d be the largest of the diameters of these paths “measured in the direction of” χ ; i. e., set

$$d = \max\{|\chi(g) - \chi(h)| \mid g, h \text{ vertices of } p_{1,r} \text{ and } r \in \mathcal{R}\}. \quad (\text{A5.3})$$

Let $b \geq d$ be a real number satisfying condition (A5.2). Then $\Delta_b(\chi)$ is a tree.

Indeed, let $\bar{p} = (v_1, v_2, \dots, v_\ell = v_1)$ be a closed path in Δ_b ; we can assume it is not reduced to a point. Pick a vertex g_j in each of the component v_j that lies outside the slice $\Gamma_\chi^{[0,b]}$ and choose, for each $j < \ell$, a path p_j that leads from g_j to g_{j+1} . The concatenation of these paths is a closed path $p = (g_1, w)$ in the Cayley graph $\Gamma(G, \mathcal{X})$. The word w is a relator of the group G ; as \mathcal{R} is a defining set of relators w is therefore freely equivalent to a product $w_1 \cdot w_2 \cdots w_h$ each subword w_j being the conjugate $u_j r_j^{\varepsilon_j} u_j^{-1}$ of an element in $\mathcal{R} \cup \mathcal{R}^{-1}$ by an \mathcal{X}^\pm -word u_j . The path $p = (g_1, w)$ is then freely equivalent to the concatenation $p_1 \cdot p_2 \cdots p_h$ of paths $p_j = (g_1, w_j)$. Each of them is a closed path based at g_1 which is made up of an initial path $p_{j,1} = (g_1, u_j)$ leading from g_1 to the point $h_j = g_1 \cdot \eta(u_j)$, the loop $p_{j,2} = (h_j, r_j^{\varepsilon_j})$ and the final path $p_{j,3} = p_{j,1}^{-1}$. Since $b \geq d$, where d is the constant given by (A5.3), the path $p_{j,2}$ runs either entirely in Γ_χ , or entirely in $\Gamma_\chi^{(-\infty, b]}$ and contains then a vertex g with $\chi(g) < 0$. The sequence $\sigma(p_{j,2})$ is therefore made up of a single vertex. It follows that the given path $\bar{p} = (v_1, v_2, \dots, v_\ell)$ in the graph $\Delta_b(\chi)$ is freely equivalent to a path \bar{p}' that is a concatenation of subpaths of the form (v_1, v_k, v_1) with $v_k = v_1$, or $v_k \in D_-$ if $v_1 \in D_+$, respectively $v_k \in D_+$ if $v_1 \in D_-$. This new path is clearly equivalent to the constant path based at v_1 . \square

We are left with establishing

LEMMA A5.6 *Assume the real number b satisfies*

$$b \geq 2 \cdot M \text{ where } M = \max\{|\chi(x)| \mid x \in \mathcal{X}\}. \quad (\text{A5.4})$$

Then every component \mathcal{E} of the intersection $\Gamma_\chi^{[0,b]} \cap \mathcal{C}_+$ contains a vertex in S_+ . (As before, \mathcal{C}_+ denotes the path component of Γ_χ that contains the vertex 1.)

Proof. Since the component \mathcal{E} is contained in \mathcal{C}_+ there exists a path $p = (1, w)$ in \mathcal{C}_+ that leads from 1 to a vertex $h \in \mathcal{E}$; let $y_1 y_2 \cdots y_\ell$ be the spelling of w . We aim at reordering the letters of w in such a way that the new word w' yields a path $p = (1, w')$ that stays inside $\Gamma_\chi^{(0,b]}$. Then the path $p' = (h, (w')^{-1})$ starts with the

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vertex $h \in \mathcal{E}$ and terminates with a vertex having χ -value 0; its endpoint then yields the desired point in S_+ .

Consider the sequence $f: \{1, 2, \dots, \ell\} \rightarrow \mathbb{R}$ of real numbers $f_j = \chi(y_j)$. By assumption, $s = \sum_{1 \leq j \leq \ell} f_j \in [0, b]$. We have to find a permutation π of $\{1, 2, \dots, \ell\}$ so that all the partial sums $s_{\pi, k} = \sum_{1 \leq j \leq k} f_{\pi(j)}$ lie in the interval $[0, b]$.

To establish the existence of π we prove a stronger assertion, namely: *for every number $c \in [0, b]$ and every sequence $f: \{1, 2, \dots, \ell\} \rightarrow \mathbb{R}$ satisfying $|f_j| \leq M$ and $c + \sum_{1 \leq j \leq \ell} f_j \in [0, b]$, there exists a permutation π such that every partial sum $c + \sum_{1 \leq j \leq k} f_{\pi(j)}$ lies in $[0, b]$.* The proof will be by induction on ℓ .

If $\ell = 0$ the claim holds. So assume $\ell > 0$. Let J_+ be the set of indices with $f_j \geq 0$ and let J_- be the complement of J_+ in the interval $J = \{1, 2, \dots, \ell\}$. Set

$$s_+ = \sum \{s_j \mid j \in J_+\} \quad \text{and} \quad s_- = \sum \{s_j \mid j \in J_-\}.$$

Two cases now arise. If $c \leq b/2$, consider $c_+ = c + s_+$. If $c_+ \leq b$, let π be a permutation which lists every $j \in J_+$ before the indices $j' \in J_-$. Then the partial sums $c + \sum_{1 \leq j \leq k} f_{\pi(j)}$ all belong to $[0, b]$. Otherwise, there exists a subset J'_+ of J_+ so that $c' = c + \sum \{f_j \mid j \in J'_+\}$ lies in the interval $]b/2, b]$. As the set J'_+ has at least one element, $J' = \{1, 2, \dots, \ell\} \setminus J'_+$ has fewer than ℓ elements and so the induction hypothesis applies to the couple (c', J') . If, on the other hand, $c_+ \geq b$ a reasoning, similar to the preceding one but with the rôles of J_+ and of J_- exchanged, allows one, either to complete the proof in a single step or to apply the induction hypothesis. \square

A5.1b Statement and proof of the local version

As mentioned in the introduction to section A5, the main result has two versions. We are now ready to establish the local version, namely

THEOREM A5.7 *Let G be a finitely generated group and assume χ is a non-zero character of G so that both $[\chi]$ and $-[\chi]$ lie outside $\Sigma^1(G)$. If G is finitely related, the kernel N of χ admits a decomposition $N = S_- \star_{S_0} S_+$ as a free product with amalgamation where S_0 has infinite index in both factors S_+ and in S_- and where both factors are infinitely generated. Moreover, N contains non-abelian free subgroups.*

Proof. Let $b \geq 0$ be a real number satisfying condition (A5.2) and let $\Delta_b(\chi)$ be the N -graph given in Definition (A5.4). This graph is connected (claim (i) of Proposition A5.5) and its quotient graph $N \backslash \Delta_b$ is a segment (claim (ii)). Let \mathcal{E}_0 be the edge, i. e., the component of the slice $\Gamma_\chi^{[0, b]}$ that contains the vertex $1 \in G$. Define S_0 to be the stabilizer of \mathcal{E}_0 and S_- and S_+ to be the stabilizers of the origin and the terminus of this edge. Since $-\chi \notin \Sigma^1(G)$, the index of S_0 in S_+ is infinite and S_+ is an infinitely generated group (by claim (iii) of the quoted proposition); since $\chi \notin \Sigma^1(G)$ it follows similarly that the index $|S_- : S_0|$ is infinite and that S_- is infinitely generated. Finally, the assumption that G is finitely related implies

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that the graph Δ_b is a tree, provided b is large enough (this is the message of point (v) of Proposition A5.5). The Bass-Serre Theory (see, e. g., [Ser03, § 4, Theorem 6]) therefore allows us to conclude that the canonical inclusions induce an isomorphism $\kappa: S_- \star_{S_0} S_+ \xrightarrow{\sim} N$.

We are left with proving that N contains a free subgroup of rank 2; this fact is widely known and can be seen as follows: pick $a \in S_+ \setminus S_0$ and $b, b' \in S_- \setminus S_0$ with $b^{-1}b' \notin S_0$ and put $x = [a, b] = aba^{-1}b^{-1}$ and $y = [a, b']$. Then the elements

$$\begin{aligned} x &= [a, b] = aba^{-1}b^{-1}, & x^{-1} &= [b, a] = bab^{-1}a^{-1}, \\ y &= [a, b'] = ab'a^{-1}(b')^{-1}, & y^{-1} &= [b', a] = b'a(b')^{-1}a^{-1} \end{aligned}$$

are in normal form and the products $xy, xy^{-1}, x^{-1}y, \dots, y^{-1}x$ and $y^{-1}x^{-1}$ admit so little of cancellation that no freely reduced word $w \neq 1$ in x, x^{-1}, y and y^{-1} can represent $1 \in N$. \square

REMARKS A5.8 a) The idea of the above proof of Theorem A5.7 goes back to a lecture Robert Bieri gave in the late 1980's; cf. the proof of Theorem 5.1 in Chapter I of [BS92]. Bieri's proof was not well-known at the time, as is indicated by a remark in section 1.6.4 of Peter Shalen's survey [Sha91].

b) Theorem A5.1, the global version of the result just proved, follows easily from the basic one. Indeed, let G be a finitely related group that contains no free subgroup of rank 2 and let χ be a non-zero character. Then its kernel N cannot be a generalized free product $S_- \star_S S_+$ where S_0 has infinite index in both N_- and in N_+ . By Theorem A5.7 at least one of the points $[\chi], -[\chi]$ must therefore lie in $\Sigma^1(G)$.

A5.2 Main result for groups of type FP_2

In Theorem A5.1 the group G is assumed to admit a finite presentation; the conclusion of the theorem need not be valid if this hypothesis is omitted (cf. remark A5.3a). The hypothesis that G be finitely related can, however, be slightly weakened: it suffices that G is of type FP_2 over a commutative ring K .

We begin by recalling the relevant definitions and some facts. Let $R \triangleleft F \xrightarrow{\pi} G$ be a free presentation of G . The abelianization R_{ab} of R is a left G -module by conjugation in F , called the *relation module* of the presentation. This module is the first term of the short exact sequence

$$R_{\text{ab}} \xrightarrow{\kappa} \mathbb{Z}G \otimes_{\mathbb{Z}F} IF \xrightarrow{\nu} IG \quad (\text{A5.5})$$

(see, e. g., [HS97, Thm. IV,6.3]). Here IF and IG denote the augmentation ideals of $\mathbb{Z}F$ and $\mathbb{Z}G$, respectively, and the group ring $\mathbb{Z}G$ in the term $\mathbb{Z}G \otimes_{\mathbb{Z}F} IF$ is viewed as a right $\mathbb{Z}F$ -module via the projection π . The map κ sends the class $f \cdot R$ to $1_{\mathbb{Z}G} \otimes (f - 1)$ and ν takes $g \otimes (f - 1)$ to the product $g \cdot (\pi(f) - 1)$.

The augmentation ideal IF of the group ring $\mathbb{Z}F$ is a free $\mathbb{Z}F$ -module whose rank equals the rank of F (see, e. g., [HS97, Th. IV.5.5]), and so the short exact

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sequence (A5.5) is a *free presentation of the augmentation ideal IG* , viewed as a left $\mathbb{Z}G$ -module.

Consider now a commutative ring K .⁴ Since the additive group of IG is free abelian, the sequence (A5.5) remains exact when tensored with $K \otimes_{\mathbb{Z}} -$. Schanuel's Lemma (see, e. g., [Bro94, Lemma VIII.4.2]) therefore implies that whether or not the KG -module $K \otimes R_{\text{ab}}$ is finitely generated does neither depend on the choice of the *finitely generated* free group F nor on the epimorphism $\pi: F \twoheadrightarrow G$.

The trivial G -module K has the free presentation

$$0 \rightarrow K \otimes IF \rightarrow KG \rightarrow K \rightarrow 0.$$

Upon splicing this sequence with the short exact sequence obtained by applying the functor $K \otimes -$ to sequence (A5.5), one arrives at a beginning of a KG -free resolution of K , namely

$$0 \rightarrow K \otimes R_{\text{ab}} \xrightarrow{1 \otimes \kappa} KG \otimes_{\mathbb{Z}F} IF \xrightarrow{\nu} KG \rightarrow K \rightarrow 0. \quad (\text{A5.6})$$

The previous considerations and Schanuel's Lemma then lead to the following

LEMMA A5.9 *For every finitely generated group G and every commutative ring K (with $1 \neq 0$), the following statements are equivalent:*

- (i) *there exists a free presentation $R \triangleleft F \twoheadrightarrow G$ with F finitely generated so that $K \otimes_{\mathbb{Z}} R_{\text{ab}}$ is a finitely generated left KG -module;*
- (ii) *for every free presentation $R_1 \triangleleft F_1 \twoheadrightarrow G$ with F_1 finitely generated the KG -modules $K \otimes_{\mathbb{Z}} (R_1)_{\text{ab}}$ is finitely generated;*
- (iii) *the group G is of type FP_2 over K .*

A5.2a A characterization of groups of type FP_2

The proof of Theorem A5.14 below makes use of a characterization of groups of type FP_2 over a commutative ring K . This result is Lemma 2.1 in [BS78], except for the fact that there groups of type FP_2 over K are called *almost finitely presented over K* .

LEMMA A5.10 *Let $R \triangleleft F \xrightarrow{\pi} G$ be a free presentation of the finitely generated group G with F free of finite rank. Then the KG -module $K \otimes R_{\text{ab}}$ is finitely generated if, and only if, there exists a finite subset $\mathcal{R}_1 \subset R$ so that the kernel M of the projection $\pi_*: G_1 = F / \text{gp}_F(\mathcal{R}_1) \twoheadrightarrow G$ has the property that $K \otimes M_{\text{ab}} = 0$.*

Proof. Let $R_1 \subseteq R$ be normal subgroup of F . The extension $R_1 \xrightarrow{\mu} R \rightarrow R/R_1$ induces then the right exact sequence

$$K \otimes_{\mathbb{Z}} (R_1 \cdot R') / R' \xrightarrow{\mu_*} K \otimes_{\mathbb{Z}} R_{\text{ab}} \rightarrow K \otimes_{\mathbb{Z}} (R/R_1)_{\text{ab}} \rightarrow 0 \quad (\text{A5.7})$$

⁴The ring K is always assumed to possess a unit element $1 \neq 0$.

of KG -modules. Assume now the KG -module $K \otimes R_{\text{ab}}$ is finitely generated over KG . Pick a finite set $\mathcal{R}_1 \subset R_1$ so that the set $\{1 \otimes r_1 R' \mid r_1 \in \mathcal{R}_1\}$ generates $K \otimes R_{\text{ab}}$. Then the embedding μ_* in the sequence (A5.7) is surjective and so $K \otimes (R/R_1)_{\text{ab}} = 0$. But R/R_1 is the kernel M of the projection $\pi_*: G_1 = F/\text{gp}_F(\mathcal{R}_1) \twoheadrightarrow G$; so we have proved that $K \otimes M_{\text{ab}} = 0$.

Conversely, assume that $R_1 \subseteq R$ is the normal closure of a finite subset \mathcal{R}_1 and $R/R_1 = \ker(G_1 = F/R_1 \twoheadrightarrow F/R)$ is such that $K \otimes (R/R_1)_{\text{ab}} = 0$. Then the map μ_* in sequence (A5.7) is an epimorphism, whence $K \otimes R_{\text{ab}}$ is generated, as a KG -module, by the finite set $\{\mu(1 \otimes r \cdot R') \mid r \in \mathcal{R}_1\}$. \square

REMARKS A5.11 a) Let G be a group that is of type FP_2 over the commutative ring K and let $\rho: K \rightarrow L$ be a homomorphism of commutative rings (taking 1_K to 1_L). Upon tensoring the exact sequence (A5.6) with $L \otimes_K -$, one obtains an exact sequence of LG -modules; this sequence shows that G is of type FP_2 over L . The preceding remark applies, in particular, to the canonical ring homomorphism $\mathbb{Z} \rightarrow L$. Moreover, if $R \triangleleft F$ is finitely generated as a normal subgroup of F , then R_{ab} is finitely generated as a F/R -module. So we have proved

LEMMA A5.12 *For every finitely generated group G the following claims hold:*

- (i) *If G is finitely presented then G is of type FP_2 over \mathbb{Z} .*
- (ii) *If G is of type FP_2 over \mathbb{Z} then G is of type FP_2 over every commutative ring K .*

b) The converse of implication (i) has been an open problem for many years. In 1997 it was settled in the negative by M. Bestvina and N. Brady in [BB97]. Their counter examples are kernels of rank 1 characters of suitably chosen right angled Artin groups.

c) The converse of (iii) is known to be false since 1980 (see [BS80, p. 464]). A counter example is provided by the soluble matrix group put forward by H. Abels in [Abe79].

EXAMPLE A5.13 Let G_1 be the multiplicative group of all matrices of the form

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{GL}_4(\mathbb{Z}[\frac{1}{2}]).$$

where \star stands for entries in the ring $\mathbb{Z}[\frac{1}{2}]$ of all dyadic rationals; in addition, the diagonal entries are required to be positive (and hence of the form 2^m with $m \in \mathbb{Z}$). According to [Abe79, p. 205], the group G_1 is finitely presented.

The centre Z of G_1 consists of all matrices with diagonal entries equal to 1, and all other entries equal to 0, except for the entry in the upper right corner which is an arbitrary element of $\mathbb{Z}[\frac{1}{2}]$. So Z is isomorphic to the additive group

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of $\mathbb{Z}[\frac{1}{2}]$. Since Z is not finitely generated as a normal subgroup of G_1 , the factor group $G = G_1/Z$ cannot be finitely presented. On the other hand, $\mathbb{F}_2 \otimes Z = 0$ and so G is of type FP_2 over the field \mathbb{F}_2 by Lemma A5.10.

A5.2b Strengthened version of Theorem A5.7

The conclusion of Theorem A5.7 remains valid if the assumption that G be finitely related is replaced by the weaker assumption that G be of type FP_2 over some commutative ring K :

THEOREM A5.14 *Let G be a finitely generated group and let χ be a non-zero character of G so that both $[\chi]$ and $-[\chi]$ lie outside $\Sigma^1(G)$. If G is of type FP_2 over some commutative ring K , the kernel N of χ admits a decomposition $N = S_- \star_{S_0} S_+$ as a free product with amalgamation where S_0 has infinite index in both factors S_- and in S_+ and where both factors are infinitely generated.*

Moreover, N contains non-abelian free subgroups.

Proof. The construction of the graphs $\Delta_b(\chi)$ and properties (i) through (iv) of Proposition A5.5 require merely that G is finitely generated. The finite presentation is only used in establishing that Δ_b is a tree if b is sufficiently large. The fact that Δ_b is a tree under the weaker hypothesis can be seen as follows:

Lemma A5.10 yields a short exact sequence of groups $M \twoheadrightarrow G_1 \xrightarrow{\pi} G$ where G_1 is finitely related and $K \otimes M_{\text{ab}} = 0$. Let $\psi = \chi \circ \pi$ be the character of G_1 induced by the given character χ and put $N_1 = \ker \psi$. Next choose a finite presentation F/R_1 of G_1 and use the basis \mathcal{X} of F as a generating system of G_1 and, via π , of G . The canonical map $\pi: G_1 \rightarrow G$ gives rise to a covering map $\pi_*: \Gamma_1 \rightarrow \Gamma$ of the corresponding Cayley graphs. This map sends the subgraph $\Gamma_{1,+} = \Gamma(G_1, \mathcal{X})_\psi$ onto the subgraph $\Gamma_+ = \Gamma(G, \mathcal{X})_\chi$ and hence each of the connected components of $\Gamma_{1,+}$ onto a component of Γ_+ . This implies, in particular, that $\pi: G_1 \rightarrow G$ maps the stabilizer $S_{1+} = (\ker \psi) \cap \mathcal{C}_{1+}$ of the component of $\Gamma_{1,+}$ containing 1_{G_1} onto the stabilizer S_+ of \mathcal{C}_+ . One verifies similarly that $\pi: G_1 \rightarrow G$ maps the stabilizer S_{1-} of \mathcal{C}_{1-} , the connected component of $\Gamma_\psi^{(-\infty, b]}$ containing 1_{G_1} , onto the stabilizer S_- of the connected component \mathcal{C}_- , and that π maps the stabilizer S_1 of the connected component $\mathcal{E}_{1,0}$ of $\Gamma_\psi^{[0, b]}$ onto the stabilizer S_0 of the edge \mathcal{E}_0 .

We are now ready to prove that the graph $\Delta_b(\chi)$ is a tree if b is suitably chosen. Recall that the character ψ is the composition $\chi \circ \pi$. Since M is the kernel of the epimorphism $\pi: G_1 \rightarrow G$, the columns of the following diagram (A5.8) are exact. Select now the real number $b \in \text{im } \psi = \text{im } \chi$ so large that $\Delta_b(G_1)$ is a tree (cf. Theorem A5.7). Then the inclusions of S_{1-} , S_{1+} and of S_1 in G_1 induce an isomorphism $S_{1-} \star_{S_1} S_{1+} \xrightarrow{\sim} N_1 = \ker \psi$.

Notes

$$\begin{array}{ccccc}
M & \xrightarrow{=} & M & & \\
\downarrow \iota_* & & \downarrow \iota & & \\
N_1 = \ker \psi & \xrightarrow{\quad} & G_1 & \xrightarrow{\psi} & \mathbb{R} \\
\downarrow \pi_* & & \downarrow \pi & & \downarrow = \\
N = \ker \chi & \xrightarrow{\quad} & G & \xrightarrow{\chi} & \mathbb{R}
\end{array} \tag{A5.8}$$

If one replaces the middle term N_1 of the *left* column of diagram (A5.8) by $S_{1-} \star_{S_1} S_{1+}$, the *first row* of diagram (A5.9) results:

$$\begin{array}{ccccc}
M & \xrightarrow{\quad} & S_{1-} \star_{S_1} S_{1+} & \xrightarrow{\quad} & N \\
\downarrow \pi_* & & \downarrow \pi_* & & \downarrow = \\
L = \ker \rho & \xrightarrow{\quad} & S_- \star_{S_0} S_+ & \xrightarrow{\quad} & N
\end{array} \tag{A5.9}$$

Its maps are all induced by the obvious inclusions and so the diagram commutes.

In a previous part of the proof we have seen that π maps the stabilizers S_{1-} , S_{1+} and of S_1 occurring in the middle term of the first row onto the corresponding stabilizers occurring in the second row. The middle vertical map is therefore surjective, whence so is the left vertical map. This fact implies that L is reduced to the neutral element. Indeed, since $K \otimes M_{\text{ab}} = \{0\}$ and as abelianization and tensor product are right exact functors, $K \otimes L_{\text{ab}} = \{0\}$. On the other hand, the epimorphism $S_- \star_{S_0} S_+ \twoheadrightarrow N$ is induced by the inclusions of the factors; so its kernel L intersects them trivially and thus acts freely on the canonical tree of $S_- \star_{S_0} S_+$, whence L is a free group. As $K \otimes L_{\text{ab}} = \{0\}$ it must be trivial.

The canonical inclusions thus induce an isomorphism $S_- \star_{S_0} S_+ \xrightarrow{\sim} N$. This can only hold if the graph Δ_b is a tree (see, e.g., [Ser03, §4, Theorem 6]). \square

Theorem A5.14 is the local version of our final result, just as Theorem A5.7 is the local version of Theorem A5.1:

THEOREM A5.15 *Let G be a finitely generated group that does not contain a non-abelian free subgroup. If G is of type FP_2 over some commutative ring $K \neq \{0\}$ then*

$$\Sigma^1(G) \cup -\Sigma^1(G) = S(G). \tag{A5.10}$$

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B Complements to the Cayley graph approach to Σ^1

In Chapter A, the invariant $\Sigma^1(G)$ has been defined via the Cayley graph $\Gamma(G, \mathcal{X})$ associated to a finite generating system $\eta: \mathcal{X} \rightarrow G$ of the group G and some of its properties have been established. There exist alternate definitions; they are the subjects of Chapters C and D. Prior to moving on to them, we discuss some further topics that can conveniently be treated in the frame-work of Cayley graphs.

In section B1 we explain how the invariants of a group and its subgroups are related. An algebraic version of the Σ^1 -criterion is then derived in section B2; it can be used to construct algorithms which find a subset of Σ^1 . Section B3 concentrates on rank 1 points in Σ^1 . According to Proposition A3.4, these points can be described in terms of ascending HNN-extensions with a finitely generated base group. In some cases, all of Σ^1 can be found with this approach.

In the final section, an algorithm for computing Σ^1 of a one-relator group is established; it is due to Ken Brown [Bro87b]. The present justification combines results from sections B2 and B3 with basic results about one-relator groups, first proved by W. Magnus in the 1930's.

B1 Computing Σ^1 via change of groups

One of the themes of this monograph is the computation of the invariant Σ^1 for various classes of groups. In so doing, it is often helpful to consider, not only the group one is primarily interested in, but also related groups, for instance subgroups or quotient groups. This section collects results that are useful in this context. It starts out with some remarks on the morphism of character spheres induced by a group homomorphism and then derives some relations between the invariants of two or more suitably related groups. As an application, we determine in section B1.3 the invariants of a graph groups.

B1.1 Morphisms between character spheres

Let $\varphi: G \rightarrow G_1$ be a homomorphism of finitely generated groups. It induces an \mathbb{R} -linear map $\text{Hom}(\varphi, \mathbb{R}): \text{Hom}(G_1, \mathbb{R}) \rightarrow \text{Hom}(G, \mathbb{R})$ sending $\chi: G_1 \rightarrow \mathbb{R}$ to $\chi \circ \varphi$. If a character $\chi: G \rightarrow \mathbb{R}$ vanishes on the subgroup $\text{im } \varphi$ of G_1 the composition $\chi \circ \varphi$ represents the zero-class $[0]$, a class that lies outside of $S(G_1)$. Accordingly we define the morphism induced by φ as follows:

$$\varphi^*: S(G_1, \text{im } \varphi)^c \rightarrow S(G), \quad \varphi^*[\chi] = [\chi \circ \varphi]. \quad (\text{B1.1})$$

B1.1a Determining the image of φ^*

It will be useful to have a good idea of the form of the fibers and of the image of the morphism φ^* . We begin by determining its image. It is a subsphere, for the image of the \mathbb{R} -linear map $\text{Hom}(G_1, \mathbb{R}) \rightarrow \text{Hom}(G, \mathbb{R})$ is a subspace and so its rays emanating from the origin form a subsphere.

One can describe this image more explicitly:

LEMMA B1.1 (i) *The homomorphism $\varphi: G \rightarrow G_1$ induces the morphism*

$$\varphi^*: S(G_1, \varphi(G))^c \twoheadrightarrow S(G, \varphi^{-1}(G'_1)) \hookrightarrow S(G) \quad (\text{B1.2})$$

(ii) *If φ is surjective, the morphism takes on the simpler form*

$$\varphi^*: S(G_1) \xrightarrow{\sim} S(G, \ker \varphi) \hookrightarrow S(G). \quad (\text{B1.3})$$

Proof. (ii) If $\varphi: G \twoheadrightarrow G_1$ is surjective the induced morphism

$$\varphi^*: \text{Hom}(G_1, \mathbb{R}) \rightarrow \text{Hom}(G, \mathbb{R}), \quad \psi \mapsto \psi \circ \varphi$$

is injective and its image is the subspace of $\text{Hom}(G, \mathbb{R})$ consisting of the characters that vanish on $\ker \varphi$. This proves assertion (ii).

(i) The homomorphism $\varphi: G \rightarrow G_1$ induces a homomorphism of abelian groups $\varphi_*: G/\varphi^{-1}(G'_1) \hookrightarrow (G_1)_{\text{ab}}$; this homomorphism is injective. As \mathbb{R} is a divisible group, hence an injective \mathbb{Z} -module, every character of $G/\varphi^{-1}(G'_1)$ extends to a character of $(G_1)_{\text{ab}}$ (see, e. g., [Rob96, 4.1,2]). The morphism

$$\varphi^*: S(G_1, \text{im } \varphi)^c \rightarrow S(G/(\varphi^{-1}(G'_1)))$$

is therefore surjective. Now use that $S(G/(\varphi^{-1}(G'_1))) \xrightarrow{\sim} S(G, \varphi^{-1}(G'_1))$ is an isomorphism (by claim (ii)). \square

B1.1b Determining the fibers of φ^*

Let $[\psi]$ be a point in the image of φ^* and let $[\chi]$ and $[\chi_1]$ two points in the fiber above $[\psi]$. Then $[\chi \circ \varphi] = [\psi] = [\chi_1 \circ \varphi]$ and so there is a positive scalar λ so that $\chi \circ \varphi = (\lambda \cdot \chi_1) \circ \varphi$. The difference $(\lambda \cdot \chi_1) - \chi$ is then in the kernel V of the \mathbb{R} -linear map $\text{Hom}(\varphi, \mathbb{R}): \text{Hom}(G_1, \mathbb{R}) \rightarrow \text{Hom}(G, \mathbb{R})$. Conversely, if $v \in V$ then $[\chi]$ and $[\chi + v]$ have the same image under φ^* . The points in the fiber above $[\psi] = \varphi^*([\chi])$ can therefore be parametrized by the characters in the *affine* subspace $\chi + V$ of $\text{Hom}(G_1, \mathbb{R})$. The fiber is therefore an open hemisphere in the sphere made up of the rays in the subspace spanned by V and χ ; in particular, the dimension of the fiber is equal to the dimension of V and the boundary of the fiber is the subsphere $S(G_1, \text{im } \varphi)$. Since the dimension of V is the torsion-free rank of $(G_1)_{\text{ab}}/(\text{im } \varphi_{\text{ab}}) \xrightarrow{\sim} G_1/(\varphi(G) \cdot G'_1)$, we have proved

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LEMMA B1.2 *The fiber of the point $\varphi^*([\chi])$ consists of the points represented by the characters in $\chi + V$ where V is the kernel of*

$$\text{Hom}(\varphi, \mathbb{R}): \text{Hom}(G_1, \mathbb{R}) \rightarrow \text{Hom}(G, \mathbb{R}).$$

It is a hemisphere in a subsphere of dimension $\dim V = r_0(G_1/(\varphi(G) \cdot G'_1))$ and it is bounded by the subsphere $S(G_1, \varphi(G))$.

The previous lemma allows one to spell out when φ induces an injective morphism:

COROLLARY B1.3 *For every homomorphism $\varphi: G \rightarrow G_1$ of finitely generated groups, the following three conditions are equivalent:*

- (i) $\varphi^*: S(G, \text{im } \varphi)^c \rightarrow S(G_1)$ is injective;
- (ii) the subgroup $\text{im } \varphi \cdot [G, G]$ has finite index in G ;
- (iii) φ^* is defined on $S(G)$.

Since the morphism $\varphi^*: S(G, \text{im } \varphi)^c \rightarrow S(G_1)$ is induced by a linear map, it is compatible with the geometric structures of the spheres; in particular, it is continuous and sends a pair of antipodal points to a pair with the same property. A further, useful fact is recorded in

LEMMA B1.4 *If $[\chi] \in S(G, \text{im } \varphi)^c$ then $\text{rk}[\chi] \geq \text{rk } \varphi^*[\chi] \geq 1$. In particular, the image of a rational point is rational.*

B1.2 Properties of Σ^1 under change of groups

In this section, we present results that relate the invariant of the given group G to invariants of related groups, such as quotient groups or subgroups of finite index.

B1.2a Isomorphisms and automorphisms of groups

We begin with a remark that is long overdue. The definition of $\Sigma^1(G)$ uses a finite generating system $\eta: \mathcal{X} \rightarrow G$. By Theorem A2.3, the choice of this system plays no rôle, a first fact that justifies calling $\Sigma^1(G)$ an invariant. But more is true: *the construction $G \mapsto \Sigma^1(G)$ assigns to isomorphic groups isomorphic invariants*. This fact is a direct consequence of the independence of Σ^1 on the choice of the generating system. Indeed, let $\alpha: G \xrightarrow{\sim} G_1$ be an isomorphism of finitely generated groups, let $\eta: \mathcal{X} \rightarrow G$ be a generating system of G and use α to push η forward to a generating system $\eta_1 = \alpha \circ \eta$ of G_1 . Then α induces an obvious graph isomorphism $\alpha_*: \Gamma(G, \mathcal{X}) \xrightarrow{\sim} \Gamma(G_1, \alpha(\mathcal{X}))$ of Cayley graphs. Moreover, if χ_1 is a non-zero character of G_1 and $\chi = \chi_1 \circ \alpha$, then α_* maps the subgraph $\Gamma(G, \mathcal{X})_\chi$ onto the subgraph $\Gamma(G_1, \alpha(\mathcal{X}))_{\chi_1}$. Consequently, α induces an isomorphism $\alpha^*: \Sigma^1(G_1) \xrightarrow{\sim} \Sigma^1(G)$.

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Note that the assignment $\alpha \mapsto \alpha^*$ is contravariant. If α is an automorphism of G , we consider therefore the assignment $\alpha \mapsto (\alpha^{-1})^*$; it yields a homomorphism of $\text{Aut}(G)$, the group of automorphisms of G , into the group of homeomorphisms of the sphere $S(G)$ under which the subset $\Sigma^1(G)$ is invariant.

For later reference we summarize the preceding discussion in

PROPOSITION B1.5 *For every finitely generated group G , the assignment*

$$\alpha \mapsto (\alpha^{-1})^*: S(G) \xrightarrow{\sim} S(G)$$

is a group homomorphism of $\text{Aut}(G)$ into the group of homeomorphisms of the sphere $S(G)$. The invariant $\Sigma^1(G)$ is stable under this action of $\text{Aut}(G)$ on $S(G)$.

Application to free soluble groups. The invariant Σ^1 provides a method for showing that finitely generated groups in certain classes of soluble groups are infinitely related. Prior to turning to this application, we record an easy consequence of Theorems A5.1 and A4.1 and the preceding proposition:

COROLLARY B1.6 *Let G be a finitely presented group with no non-abelian free subgroups. If G admits an automorphism inducing $-\mathbb{1}$ on $G/\sqrt{G'}$ then $\Sigma^1(G) = S(G)$ and so G' is a finitely generated group.*

Proof. If α induces $-\mathbb{1}$ on $G/\sqrt{G'}$ then $\Sigma^1(G) = (\alpha^{-1})^*(\Sigma^1(G)) = -\Sigma^1(G)$. On the other hand, Theorem A5.1 guarantees that $S(G) = \Sigma^1(G) \cup -\Sigma^1(G)$. So $\Sigma^1(G) = S(G)$, whence G' is finitely generated by Theorem A4.1. \square

The corollary can be applied to relatively free groups in a variety \mathcal{V} that is distinct from the variety of all groups (see [Neu67, Section I.3] for terminology and results relevant to the present context). If \mathcal{V} is a soluble variety, Corollary B1.6, in conjunction with a result of J. R. J. Groves [Gro71], yields the next result (see [Str84, Thm. 8], cf. [BS78, Cor. B1]).

COROLLARY B1.7 *A finitely generated relatively free group of a soluble variety \mathcal{V} admits a finite presentation if, and only if, it is nilpotent-by-finite.*

Proof. By Theorem A(ii) in [Gro71] every soluble variety either contains a metabelian subvariety of the form $\mathcal{A}_p\mathcal{A}$, or it is contained in a product variety of the form $\mathcal{N}_c\mathcal{B}_e$. Here $\mathcal{A}, \mathcal{A}_p$ are the varieties respectively, of abelian groups and elementary abelian p -groups, while \mathcal{N}_c denotes the variety of all nilpotent groups of class at most c and \mathcal{B}_e is the Burnside variety of exponent e .

Suppose \mathcal{V} contains $\mathcal{A}_p\mathcal{A}$ for some prime p and G is a non-cyclic, finitely generated, relatively free group of \mathcal{V} with basis \mathcal{B} . Then $G/(G'' \cdot (G')^p)$ is a non-cyclic $\mathcal{A}_p\mathcal{A}$ -free group and so the derived group of G is infinitely generated. Since the function which assigns to each basis element in \mathcal{B} its inverse, extends to an automorphism of G inducing $-\mathbb{1}$ on G_{ab} , the previous corollary forces G to be infinitely

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related. A relatively free group G in \mathcal{V} admitting a finite presentation is thus either cyclic, or \mathcal{V} is contained in a product variety of the form $\mathcal{N}_c\mathcal{B}_e$, whence G is nilpotent-by-finite. The converse is clear. \square

B1.2b Epimorphisms of groups

The morphism of spheres induced by a group homomorphism $\varphi: G \rightarrow G_1$ is, in general, only defined on a proper subset of $S(G_1)$. The situation improves if the homomorphism is surjective or if is the inclusion of a subgroup with finite index (see Corollary B1.3). In both cases, a simple formula relates then the complements of the invariants to each other.

The case of an epimorphism has already been treated in Proposition A4.5. We restate this result as

COROLLARY B1.8 *If $\pi: G \twoheadrightarrow Q$ be an epimorphism of finitely generated groups, the induced morphism π^* maps the sphere $S(Q)$ bijectively onto the subsphere $S(G, \ker \varphi)$ of $S(G)$ and the inclusion $\pi^*(\Sigma^1(Q)^c) \subseteq \Sigma^1(G)^c$ holds.*

If $\ker \pi$ is finitely generated qua group this inclusion can be sharpened to the equality $\pi^(\Sigma^1(Q)^c) = \Sigma^1(G)^c \cap S(G, \ker \varphi)$.*

EXAMPLE B1.9 Let G be the direct internal product of two normal subgroups G_1 and G_2 and let $\pi_i: G \twoheadrightarrow G_i$ be the canonical projection with kernel G_j where $\{i, j\} = \{1, 2\}$. If the corollary is applied to the projections π_1 and π_2 the equalities

$$\Sigma^1(G)^c \cap S(G, G_2) = \pi_1^*(\Sigma^1(G_1)^c) \quad \text{and} \quad \Sigma^1(G)^c \cap S(G, G_1) = \pi_2^*(\Sigma^1(G_2)^c)$$

result. It follows that $\Sigma^1(G)^c$ contains $\pi_1^*(\Sigma^1(G_1)^c) \cup \pi_2^*(\Sigma^1(G_2)^c)$. Proposition A2.7 yields a better result: it shows that this inclusion is an equality

Applications to infinitely related soluble groups. Assume G is a finitely generated soluble group and $Q = G/N$ is a factor group of G . It may happen that G admits a finite presentation, but G/N does no longer have this property (cf. Abels' group mentioned in example A5.13). If, however, Q can be shown to be infinitely related with the help of Theorem A5.1, then G will be infinitely related, too. One has:

COROLLARY B1.10 *Let G be a finitely generated group with no non-abelian free subgroups. If Q is a quotient group of G and if $\Sigma^1(Q)^c$ contains a pair of antipodal points, then G is infinitely related.*

Proof. Assume $\pi: G \twoheadrightarrow Q$ is an epimorphism and $\psi: Q \rightarrow \mathbb{R}$ and $-\psi$ represent points of $\Sigma^1(Q)^c$. By Corollary B1.8 the character $\chi = \psi \circ \pi$ and its negative represent points in $\Sigma^1(G)^c$ and so Theorem A5.1 permits us to conclude that G does not admit a finite presentation. \square

B1.2c Subgroups of finite index

In computing the invariant of a group it is sometimes useful to pass to a subgroup of finite index. The following result provides the link between the invariants of the group and its subgroup:

PROPOSITION B1.11 *Suppose G is finitely generated and $\iota: H \hookrightarrow G$ is the inclusion of a subgroup of finite index. Then the induced morphism ι^* maps the sphere $S(G)$ bijectively onto the subsphere $S(H, H \cap G')$ and the following formula holds:*

$$\iota^*(\Sigma^1(G)^c) = \Sigma^1(H)^c \cap S(H, H \cap G'). \quad (\text{B1.4})$$

Proof. The first claim follows from Lemma B1.1 and Corollary B1.3. The second will be proved by going back to the Cayley-graph definition of the invariant and exploiting the independence of the invariant on the chosen generating system.

Let $\mathcal{X} \subset G$ be a finite set generating G . Given $\chi: G \rightarrow \mathbb{R}$, find a subset $\mathcal{T} \subset G$ that contains the unit element 1, represents the homogeneous space $H \backslash G$ and satisfies $\chi(\mathcal{T}) \leq 0$. For $t \in \mathcal{T}$ and $x \in \mathcal{X}$ let \overline{tx} denote the element of \mathcal{T} representing $H \cdot tx$. Then the family $\mathcal{Y} = \{t \cdot x \cdot \overline{tx}^{-1} \mid t \in \mathcal{T} \text{ and } x \in \mathcal{X}^\pm\}$ generates the subgroup H (see, e. g., [Hal76, Lemma 7.2.2]).

Assume now that $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$ is connected. For each h in $H \cap G_\chi$ there exists then a path $p = (1, w)$ in Γ_χ from 1 to h . If w has the spelling $x_1 x_2 \dots x_m$ with letters in \mathcal{X}^\pm , let u_w denote the word $(1 \cdot x_1 t_1^{-1}) \cdot (t_1 x_2 t_2^{-1}) \cdots (t_{m-1} x_m t_m^{-1}) \cdot t_m$, where $t_i = \overline{t_{i-1} \cdot x_i}$ for each i . Since $h \in H$ and as $1 \in \mathcal{T}$, one has $t_m = 1$ and so u is actually a \mathcal{Y} -word. Moreover, if $y_1 y_2 \dots y_m$ is the \mathcal{Y} -spelling of u then

$$\chi(y_1 \dots y_i) = \chi(x_1 \dots x_i \cdot t_i^{-1}) \geq \chi(x_1 \dots x_i) \geq 0$$

for each i . This means that the path $(1, y_1 \dots y_m)$ runs in the subgraph $\Gamma(H, \mathcal{Y})_{\chi|H}$. It follows that $\Gamma(G, \mathcal{Y})_{\chi|H}$ is connected and so $\chi|H$ represents a point of $\Sigma^1(H)$.

Conversely assume that $[\chi|H] \in \Sigma^1(H)$. Let $\eta: \mathcal{Y} \rightarrow H$ of H be a finite generating system of H and choose, as before, a subset \mathcal{T} of G that contains 1, represents $H \backslash G$ and satisfies $\chi(\mathcal{T}) \leq 0$. Then $\mathcal{Y} \cup \mathcal{T}$ will generate G .

We claim the graph $\Gamma_\chi = \Gamma(G, \mathcal{Y} \cup \mathcal{T})_\chi$ is connected. This graph contains the subgraph $\Gamma(G, \mathcal{Y})_{\chi|H}$ which is connected by hypothesis. Moreover, given a vertex $g \in \Gamma_\chi$ there exists $h \in H$ and $t \in \mathcal{T}$ with $g = h \cdot t$. Then $\chi(h) = \chi(g \cdot t^{-1}) \geq \chi(g) \geq 0$ and so there exists a path $p = (1, w(\mathcal{Y}^\pm))$ in the subgraph $\Gamma(G, \mathcal{Y})_{\chi|H}$ from 1 to h . The path $(1, w(\mathcal{Y}^\pm) \cdot t)$ then leads inside Γ_χ from 1 to g . \square

REMARK B1.12 An analogue of the finite index theorem is used in [BS80] for the invariant $\Sigma^0(Q; A)$ with Q a fg abelian group. The aim there is to simplify the construction of a finite presentation of a metabelian group M with quotient G by passing from the abelian quotient G of M to a free abelian quotient H of a subgroup of M .

I wonder whether there are situations where Proposition B1.11 can be applied in the way indicated by

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PROBLEM B1.13 Find situations where one is interested in the invariant $\Sigma^1(G)$ a group G that admits a subgroup of finite index which is easier to deal with than G itself and for which Σ^1 can be computed.

The only classes I'm actually aware of are the class of abelian groups and that of co-compact Fuchsian groups of positive genus. (The groups of the second class have surface groups of positive genus as subgroups with finite index.)

B1.2d Joins of subgroups

Let G be a finitely generated group which is generated by two finitely generated subgroups G_1 and G_2 , say. Familiar examples of this situation are a) the free product $G_1 \star G_2$ and b) the free product with amalgam $G_1 \star_A G_2$. We aim at finding a useful condition that guarantees that a non-zero character $\chi: G \rightarrow \mathbb{R}$ represents a point of $\Sigma^1(G)$ if its restrictions to G_1 and to G_2 are non-zero and represent points in $\Sigma^1(G_1)$ and in $\Sigma^1(G_2)$, respectively. Example a) reveals that some extra condition is needed; indeed the invariant of a free product $G_1 \star G_2$ with factors distinct from the trivial group is empty (see Example 3 in section A2.1a).

One such extra condition is spelled out in

LEMMA B1.14 Assume G is a finitely generated group which is generated by two finitely generated subgroups G_1 and G_2 . Then the implication

$$\chi(G_1 \cap G_2) \neq \{0\}, [\chi|_{G_1}] \in \Sigma^1(G_1), [\chi|_{G_2}] \in \Sigma^1(G_2) \implies [\chi] \in \Sigma^1(G) \quad (\text{B1.5})$$

is valid for every non-zero character χ of G .

Proof. For $i = 1, 2$, let $\eta_i: \mathcal{X}_i \rightarrow G_i$ be a finite generating system of G_i and let Γ denote the Cayley graph $\Gamma(G, \mathcal{X}_1 \cup \mathcal{X}_2)$. For every $g \in G_\chi$, there exists a sequence

$$(g_{1,1}, g_{2,1}, g_{1,2}, g_{2,2}, \dots, g_{1,m}, g_{2,m}) \text{ with } g_{1,j} \in G_1 \text{ and } g_{2,j} \in G_2 \quad (\text{B1.6})$$

whose product $g_{1,1} \cdot g_{2,1} \cdots g_{1,m} \cdot g_{2,m}$ is equal to g . We aim at proving by induction on m that there exist a path p which leads inside G_χ from 1 to g .

The existence of such a path is trivial for $m = 0$; as this case tells one little about the ideas involved in the inductive step, we consider next the case $m = 1$. Then $g = g_1 \cdot g_2$ with $g_i \in G_i$. Since χ does not vanish on the intersection $H = G_1 \cap G_2$, there exist elements h_1, h_2 in H so that $g'_1 = g_1 \cdot h_1$, as well as $g'_2 = h_1^{-1} \cdot g_2 h_2$ and h_2 have non-negative χ -values. Since $g'_1 \in G_1$ and $\Gamma(G_1, \mathcal{X}_1)_{\chi|_{G_1}}$ is connected, there exists a path $p_1 = (1, w_1)$ in $\Gamma(G_1, \mathcal{X}_1)_{\chi|_{G_1}}$ that leads from 1 to g'_1 . Similarly, there exists a path $p_2 = (1, w_2)$ that connects 1 to g'_2 in $\Gamma(G_2, \mathcal{X}_2)_{\chi|_{G_2}}$. The concatenated path $p_{12} = (1, w_1 w_2)$ connects then the vertex 1 with the vertex $g'_1 \cdot g'_2$ and it stays in the subgraph Γ_χ . As h_2 is a vertex of $\Gamma(G_1, \mathcal{X}_1)_{\chi|_{G_1}}$, there exists, finally, a path $p_3 = (1, w_3)$ from 1 to h_2 in $\Gamma(G_1, \mathcal{X}_1)_{\chi|_{G_1}}$. The path $p = (1, w_1 w_2 w_3^{-1})$ leads then from 1 to $g'_1 g'_2 h_2^{-1} = g_1 h_1 \cdot h^{-1} g_2 h_2 \cdot h_2^{-1} = g_1 g_2 = g$

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without leaving Γ_χ . This latter claim follows from formulae (A2.7):

$$\begin{aligned} v_\chi(w_1 w_2 w_3^{-1}) &= \min \{v_\chi(w_1), \chi(g'_1) + v_\chi(w_2), \chi(g'_1 g'_2) + v_\chi(w_3^{-1})\} \\ &= \min \{0, \chi(g'_1) + 0, \chi(g'_1 g'_2) + \chi(h_2^{-1}) + v_\chi(w_3)\} \\ &= 0. \end{aligned}$$

The inductive step follows the same pattern as the case just treated. Assume g is the product of the sequence (B1.6) with $2m$ factors. Let g' be the product of the first $2m - 2$ factors. There exists elements h_0, h_1, h_2 in $H_{\chi|H}$ so that

$$\chi(g' h_0) \geq 0, \quad g'_1 = h_0^{-1} g_{1,m} h_1 \in (G_1)_{\chi|G_1}, \quad g'_2 = h_1^{-1} g_{2,m} h_2 \in (G_2)_{\chi|G_2}.$$

By the inductive hypothesis one can find a path $p' = (1, w')$ which leads from 1 to $g' h_0$ and stays in Γ_χ . In addition, one sees as before that there exists paths p_1, p_2 and p_3 enjoying the following properties:

- $p_1 = (1, w_1)$ runs from 1 to g'_1 in $\Gamma(G_1, \mathcal{X}_1)_{\chi|G_1}$,
- $p_2 = (1, w_2)$ runs from 1 to g'_2 in $\Gamma(G_2, \mathcal{X}_1)_{\chi|G_2}$,
- $p_3 = (1, w_3)$ runs from 1 to h_2 in $\Gamma(G_1, \mathcal{X}_1)_{\chi|G_1}$.

The concatenated path $p = (1, w' w_1 w_2 w_3^{-1})$ then leads inside Γ_χ from 1 to g . \square

A useful generalization of the previous lemma is

PROPOSITION B1.15 *Assume G is generated by a finite collection $\{G_v \mid v \in V\}$ of finitely generated subgroups. Given a non-zero character $\chi: G \rightarrow \mathbb{R}$, let $\mathcal{G}(\chi)$ denote the combinatorial graph with vertex set V and edge set the set of those pairs $\{u, v\}$ for which χ is non-zero on the intersection $G_u \cap G_v$. If the conditions*

- (i) *for every $v \in V$, the restriction of χ to G_v represents a point of $\Sigma^1(G_v)$,*
- (ii) *the graph $\mathcal{G}(\chi)$ is connected*

are satisfied, the character χ represents a point of $\Sigma^1(G)$.

Proof. We argue by induction on $n = \text{card } V$. If $n = 2$, the claim holds by Lemma B1.14; so assume $n > 2$. Choose a spanning tree T of $\mathcal{G}(\chi)$ and then a vertex v_0 of degree 1 in T . Let \mathcal{G}_1 denote the subgraph of $\mathcal{G}(\chi)$ obtained by omitting v_0 and all the edges incident with it; it is connected. Define $G_1 \subseteq G$ to be the subgroup generated by the subgroups G_v with $v \neq v_0$. Then $\mathcal{G}(\chi|G_1)$ coincides with \mathcal{G}_1 , is therefore connected, and so $[\chi|G_1] \in \Sigma^1(G_1)$ by the induction hypothesis. The claim now follows from Lemma B1.14, applied with G_1 as defined before and $G_2 = G_{v_0}$. \square

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B1.3 Application to graph groups

The notion of a graph group (or right angled Artin group) has been introduced in section A4.3b. Such a group is given by a finite combinatorial graph

$$\Delta = (\mathcal{X} = \{x_1, \dots, x_n\}, E(\Delta));$$

the graph encodes a presentation of the group, namely

$$G_\Delta = \langle x_1, x_2, \dots, x_n \mid x_j x_\ell = x_\ell x_j \text{ for every edge } \{x_j, x_\ell\} \in E(\Delta) \rangle. \quad (\text{B1.7})$$

In this section we establish a formula, announced in A4.3b, for the complement of the invariant $\Sigma^1(G_\Delta)$ and illustrate it by some examples.

B1.3a Determination of the invariant

The determination will rely on Proposition B1.15 and on the following

LEMMA B1.16 *If Δ' is a full subgraph of the graph Δ , the following assertions hold:*

- (i) *the group $G_{\Delta'}$ is a retract of G_Δ ;*
- (ii) *if Δ' is not connected, $G_{\Delta'}$ is a free product of two non-trivial subgroups and so its invariant is empty.*

Proof. (i) For every full subgraph Δ' of Δ the assignments

$$\pi(x_j) = x_j \text{ if } x_j \in V(\Delta'), \text{ and } \pi(x_j) = 1 \text{ otherwise,}$$

extend to a homomorphism π of G_Δ onto $G_{\Delta'}$. As π is the identity on the generators of $G_{\Delta'}$ it is a retract of G_Δ .

(ii) Suppose Δ' be the union of two full, disjoint and non-empty subgraphs Δ'_1 and Δ'_2 . By (i) the canonical maps $\iota_i: G_{\Delta'_i} \rightarrow G_{\Delta'}$ are embeddings and the presentation of $G_{\Delta'}$ shows that $G_{\Delta'}$ is the free product of these subgroups. Since their abelianizations are free-Abelian of rank $\text{card}(V(\Delta'_i))$, none of them is reduced to the identity. In view of Example 3 in section A2.1a) the invariant of $G_{\Delta'}$ is therefore empty. \square

One can ask for two types of depictions of $\Sigma^1(G_\Delta)$, a local and a global one. The local description looks for a procedure that allows one to decide, given a specific non-zero character χ , whether this character represents a point of the invariant. The global description seeks a formula for the entire invariant. The next theorem gives answers to both these questions.

The following terminology will be useful: a non-zero character χ determines a non-empty subset $\{x_j \in V(\Delta) \mid \chi(x_j) \neq 0\}$; define $\mathcal{L}(\chi)$ to be the full subgraph on this set and call it the *living subgraph* of χ ; call $\mathcal{L}(\chi)$ *dominating* if there exists, for every $x_\ell \in \Delta \setminus \mathcal{L}(\chi)$, a vertex $x_j \in \mathcal{L}(\chi)$ so that $\{x_j, x_\ell\}$ is an edge of Δ . A subset \mathcal{S} of $V(\Delta)$ is called *separating* if its removal results in a disconnected subgraph of Δ .

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THEOREM B1.17 *Let $G = G_\Delta$ be a right angled Artin group with graph Δ and let χ denote a non-zero character of G . If Δ is a complete graph the group G is free-Abelian of rank $\text{card } V(\Delta)$ and $\Sigma^1(G) = S(G)$. Otherwise, the following assertions are valid:*

- (i) $[\chi] \in \Sigma^1(G)$ if, and only if, the living subgraph $\mathcal{L}(\chi)$ is connected and dominating;
- (ii) the complement of $\Sigma^1(G)$ is the union of subspheres

$$\bigcup_{\mathcal{S}} S(G, \text{gp}(\mathcal{S})) \quad (\text{B1.8})$$

where \mathcal{S} runs over the minimal separating subsets of $V(\Delta)$.

REMARKS B1.18 a) Claim (i) in Theorem B1.17 is due to J. Meier and L. Van Wyck ([MV95, Theorem 4.1]) and, independently, to H. Meinert ([Mei95, Theorem 1]), while assertion (ii) seems to have been observed first by S. Papadima and A. Sugiur (see Theorem 5.5 and Proposition 5.8 in [PS06]).

b) There are at least two reasons for isolating, in the statement of Theorem B1.17, the complete graphs from the other graphs. From the point of view of the theory of groups, a right-angled Artin group contains free subgroups of rank 2, and so is highly non-commutative, unless its graph is complete. Indeed, the subgroup $H = \text{gp}(\{x_j, x_\ell\})$ of a right-angled Artin group G_Δ generated by two non-adjacent vertices is free of rank 2. Note that H is not only a subgroup, but a retract of G (see Lemma B1.16).

A second reason lies in the connectivity properties of complete graphs. In order to explain, I recall a few definitions (see [Die10, Section 1.4]). A (combinatorial) graph Δ is called *connected* if it is *non-empty* and if any two of its vertices can be linked by a path; it is called *k-connected* if it has at least $k + 1$ vertices and if the removal of any subset \mathcal{S} with at most $k - 1$ elements does not separate Δ ; in particular, a graph Δ is 1-connected if, and only if, it is connected and has at least 2 vertices. The maximum of the integers $k \geq 1$ for which Δ is *k-connected*, is called the *connectivity* of Δ and written $\kappa(\Delta)$.

According to these definitions, the connectivity of the complete graph K_n is $n - 1$ for $n \geq 2$. At first sight, this may look strange; one of its benefits is that Menger's theorem (see, e. g., [Die10, Theorem 3.3.6]) can be stated very simply in this terminology: *a graph Δ with at least 2 vertices is k-connected if, and only, any two of its vertices can be linked by k independent paths.*¹

Proof. (i) Assume first the living subgraph $\mathcal{L}(\chi)$ is connected and that it dominates Δ . Let V denote the set of *edges* of the graph Δ and, for every edge $e = \{x_i, x_j\} \in V$ set $G_e = \text{gp}(x_i, x_j)$. Then each G_e is free abelian of rank 2 and so $\Sigma^1(G_e) = S(G_e)$ (cf. Example A2.5a). But if so, the restriction of χ to G_e represents a point of G_e if e is an edge of the living graph $\mathcal{L}(\chi)$ or if e is an edge connecting a vertex

¹Two paths from v to v' are called independent if they have no interior vertices in common.

outside $\mathcal{L}(\chi)$ to a vertex in $\mathcal{L}(\chi)$. The assumption that $\mathcal{L}(\chi)$ be connected and dominating thus implies that the auxiliary graph $\mathcal{G}(\chi)$ (defined in the statement of Proposition B1.15) is connected, whence $[\chi] \in \Sigma^1(G)$ by that proposition.

Assume next that $\Delta' = \mathcal{L}(\chi)$ is *not connected*. Then $Q = G_{\Delta'}$ is a non-trivial free product (by assertion (ii) of Lemma B1.16) and so $[\chi'] = [\chi|Q]$ lies outside $\Sigma^1(Q)$, whence Corollary B1.8 allows one to conclude that $[\chi] \in \Sigma^1(G_{\Delta})^c$. If, finally, $\Delta' = \mathcal{L}(\chi)$ is *not dominating*, there exists a vertex $v \in \Delta \setminus \mathcal{L}(\chi)$ so that the full subgraph Δ_v on $\{v\} \cup V(\mathcal{L}(\chi))$ is not connected. The group $Q_v = G_{\Delta_v}$ is then a non-trivial free product and a quotient of G_{Δ} , whence it follows as before that $[\chi] \in \Sigma^1(G_{\Delta})^c$.

(ii) Let χ be a non-zero character. Assume first that $[\chi] \in S(G, \text{gp}(\mathcal{S}))$ for some separating subset \mathcal{S} of $V(\Delta)$. Then $[\chi]$ is the pullback of a character $[\chi']$ along the obvious projection $\pi: G \twoheadrightarrow G_{\Delta \setminus \mathcal{S}}$ and the subgraph $\Delta \setminus \mathcal{S}$ is not connected. Lemma B1.16 and Corollary B1.8 then show that $[\chi]$ is in $\Sigma^1(G)^c$.

Suppose now that $[\chi]$ lies outside of every subsphere $S(G, \text{gp}(\mathcal{S}))$ defined by a separating subset \mathcal{S} ; put differently, suppose that $\chi(\mathcal{S}) \neq \{0\}$ for every separating subset \mathcal{S} of Δ . Consider the subset $\mathcal{S}' = V(\Delta) \setminus \mathcal{L}(\chi)$. Then $\chi(\mathcal{S}') = \{0\}$ by the definition of $\mathcal{L}(\chi)$, so \mathcal{S}' does not separate Δ , whence $\mathcal{L}(\chi)$ is connected. One sees similarly that every set containing $V(\Delta) \setminus \mathcal{L}(\chi)$ is connected. The living graph $\mathcal{L}(\chi)$ is thus connected and dominating, and so $[\chi] \in \Sigma^1(G)$ by part (i).

The previous reasoning proves that $\Sigma^1(G)^c$ coincides with the union of all subspheres $S(G, \text{gp}(\mathcal{S}))$ where \mathcal{S} ranges over the separating subsets. But as the subsphere $S(G, \text{gp}(\mathcal{S}))$ contains the subsphere $S(G, \text{gp}(\mathcal{S}'))$ if \mathcal{S} is a subset of \mathcal{S}' , this union is not altered if \mathcal{S} ranges only over the *minimal* separating subsets. \square

B1.3b Finding the list of minimal separating subsets

Let Δ denote a *connected*, finite combinatorial graph that is not complete and let $G = G_{\Delta}$ be the associated right-angled Artin group. According to Theorem B1.17 the complement $\Sigma^1(G)^c$ of the invariant is a finite union of great subspheres $S(G, \text{gp}(\mathcal{S}))$ where \mathcal{S} ranges over the *minimal, separating* subsets of the vertex set $V(\Delta)$. This description is very concise and handy; the list L_{Δ} of minimal separating subset \mathcal{S} can, however, be rather large and, worse, it may be difficult to determine it explicitly. The following example illustrates the first difficulty.

EXAMPLE B1.19 For every positive integer $m \geq 2$, let V_m denote the set

$$\{v_0, v_{1,1}, \dots, v_{1,m}, v_{2,1}, \dots, v_{2,m}, v_3\}$$

with $2m + 2$ vertices. Define E_m to be set consisting of the $3m$ edges

$$\{v_0, v_{1,j}\}, \quad \{v_{1,j}, v_{2,j}\}, \quad \{v_{2,j}, v_3\} \text{ with } j \in \{1, 2, \dots, m\}$$

and set $\Delta_m = (V_m, E_m)$. One can visualize Δ_m as a bundle of m rods of length 3 which are tied together at both ends. The graph Δ_m has $2^m + 2m + 1$ minimal

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separating subsets. Indeed, every subset \mathcal{S} that contains exactly one inner point $v_{1,j}$ or $v_{2,j}$ of each rod, but none of the end points, is separating and minimal; there are 2^m such subsets. In addition, $\{v_0, v_3\}$ is a minimal separating subset, as are the subsets $\{v_0, v_{2,j}\}$ and $\{v_{1,j}, v_3\}$ for every index j . Consider now a non-empty subset \mathcal{S}' that is distinct from the $2^m + 2m + 1$ minimal separating subsets found so far. If it contains none of the end points v_0, v_3 and omits no inner points of some rod, it does not separate; if it contains none of the end points, but both inner points of some rod, it is not minimal. Similarly, one finds that \mathcal{S}' is non-separating or not minimal, if it contains one or two end points.

These calculations show that

$$\text{card}(L_{\Delta_m}) = 2^m + 2m + 1 > (\sqrt{2})^{2m} = \frac{1}{2} (\sqrt{2})^{\text{card}(V(\Delta_m))}.$$

We now turn to the second problem, that of finding the list of all minimal separating subsets. The next lemma can help one in solving it.

LEMMA B1.20 *For every subset \mathcal{S} of the vertex set $V(\Delta)$ of a connected, finite combinatorial graph Δ the following statements are equivalent :*

- (i) \mathcal{S} is a minimal separating subset,
- (ii) let $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ be the connected components of the subgraph induced by the complement $V(\Delta) \setminus \mathcal{S}$ of \mathcal{S} . Then $\ell \geq 2$ and every vertex $v \in \mathcal{S}$ is linked to each component \mathcal{C}_j by an edge.

Proof. Let $\mathcal{S} \subset V(\Delta)$ be a non-empty, proper subset and let $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ denote the connected components of the subgraph induced by $V(\Delta) \setminus \mathcal{S}$. Then \mathcal{S} is separating if, and only if, $\ell \geq 2$. Assume now $\ell \geq 2$ and fix a vertex $v \in \mathcal{S}$. If \mathcal{S} is minimal, the complement of $\mathcal{S} \setminus \{v\}$ must be connected and so there exists, for every component \mathcal{C}_j , a vertex v_j that is linked to v by an edge. Conversely, if every $v \in \mathcal{S}$ has the stated property, \mathcal{S} is minimal and so statement (i) holds. \square

The previous lemma allows one to replace the search for minimal separating subsets of a graph Δ by a search for full, connected subgraphs. This simplifies the task in two ways: a minimal separating subset \mathcal{S} is an unstructured object whose cardinality can *a priori* be close to that of the vertex set $V(\Delta)$ whereas one of the connected components of $\Delta \setminus \mathcal{S}$ has at most $(\text{card}(V(\Delta))/2)$ vertices. So one can proceed like this: one constructs, for each $m \geq 1$, the full, connected subgraphs of Δ with m vertices, determines their boundaries and then checks whether these boundaries fulfill statement (ii) of Lemma B1.20.

EXAMPLE B1.21 Let Δ be the 1-skeleton of a regular dodecahedron; it has 20 vertices and 30 edges. Our aim is to find the *list of all connected subgraphs* \mathcal{C} with the following properties:

- (i) the boundary $\delta(\mathcal{C})$ of \mathcal{C} is a minimal separating subset,

Notes

(ii) none of the connected components \mathcal{C}' of $\Delta \setminus \delta(\mathcal{C})$ has fewer vertices than \mathcal{C} .

It will turn out that $\Delta \setminus \delta(\mathcal{C})$ has always two components. In Figures B.1 through B.5, the chosen connected component \mathcal{C} is displayed in red, the derived component in blue.

(i) The connected subgraphs with $m \leq 3$ vertices are singletons, edges or lines of length 2. They give rise to 20 minimal separating subsets with 3 vertices, 30 such subsets with 4 vertices and 60 subsets with 5 vertices (see Figure B.1).

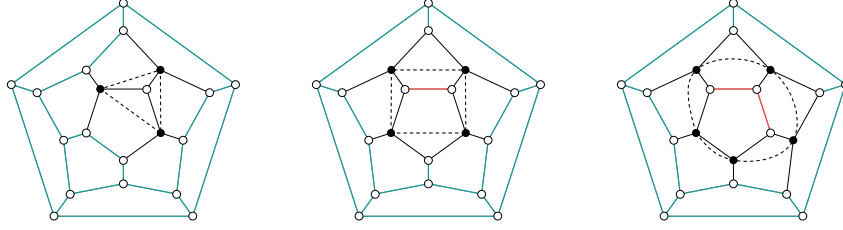


Figure B.1: Minimal separating sets of a dodecahedral graph for $m = 1, 2$ and 3

(ii) If $m = 4$, the component \mathcal{C} is a line or a tripod; in the first case, the boundary has either 5 or 6 vertices, in the second it has 6 vertices. The number of boundaries is $60 + 60 + 20 = 140$ separating subsets (see Figure B.2).

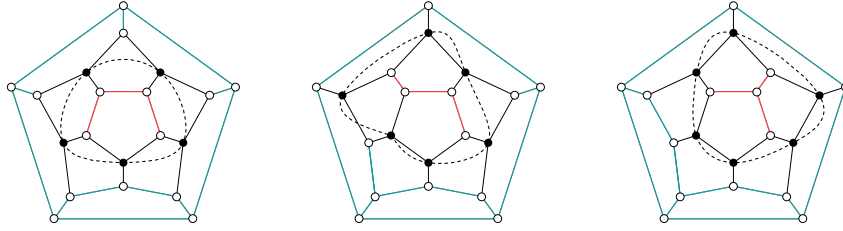
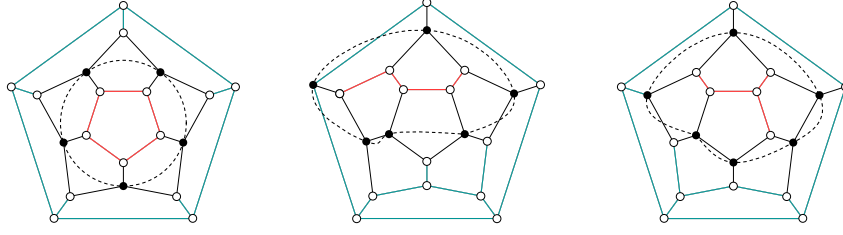
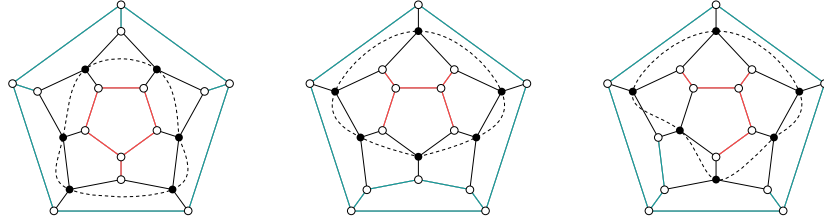


Figure B.2: Minimal separating sets of a dodecahedral graph for $m = 4$

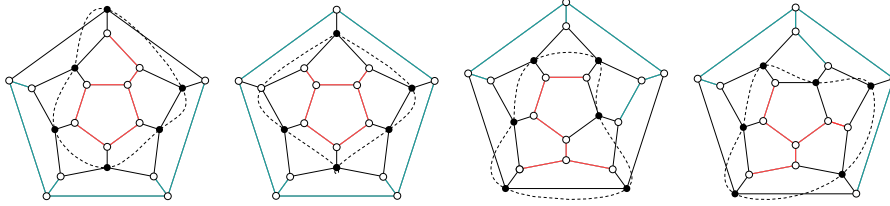
(iii) If $m = 5$, the component \mathcal{C} is a pentagon, a line not contained in a single face, or a tripod. The first case leads to 12 minimal separating subsets, each with 5 vertices. The second case gives rise to $12 \cdot 5 \cdot 3 = 180$ separating subsets with 6 vertices. Finally, there are $20 \cdot 6 = 120$ tripods; their boundaries have 6 vertices. See Figure B.3.

(iv) For $m = 6$, the components can again take on three shapes: a pentagon with a spike, a capital letter H and a tripod. The first leads to $12 \cdot 5 = 60$ separating subsets, the second to 30 such subsets, the third to $20 \cdot 3 \cdot 3 = 180$ separating subsets. All separating subsets have 6 vertices; see Figure B.4.

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Figure B.3: Minimal separating sets for $m = 5$ Figure B.4: Minimal separating sets for $m = 6$

(v) The case $m = 7$ is more involved. Detailed analysis shows that the components can be of six forms; they give rise to four types of separating subsets. In Figure B.5, the first components are a pentagon with a long handle, a pentagon with two adjacent spikes, a tripod with a long handle and a symmetric tripod. The second component of the pentagon with a long handle is different from the first component, as is the the second component of the tripod with a long handle, while both components are isomorphic in the other two cases. The separating subsets have always 6 vertices. Their numbers are: $12 \cdot 10 = 120$ for the pentagon with the long handle, $\frac{1}{2}12 \cdot 5 = 30$ for the pentagon with two spikes, $20 \cdot 3 \cdot 2 \cdot 2 = 240$ for the tripod with a long handle and $\frac{1}{2}20 \cdot 2 = 20$ for the symmetric tripod.

Figure B.5: Minimal separating sets for $m = 7$

All taken together, there are $20 + 30 + 60 + 140 + 312 + 270 + 410 = 1242$ separating subsets, a quite impressive number.

Notes

B2 The Σ^1 -criterion revisited

The techniques for computing Σ^1 discussed so far can be summarized thus. The invariant $\Sigma^1(G)$ is defined in terms of the Cayley graph $\Gamma(G, \mathcal{X})$ of a group G with respect to a finite system of generators \mathcal{X} . The computation is therefore straightforward if the Cayley graph has a simple geometric form and/or if it is made of well understood Cayley graphs with the help of suitable constructions. Cayley graphs with simple geometric form are provided by free abelian groups, and by non-abelian free groups or, more generally, free products. In the first case, $\Sigma^1(G) = S(G)$, in the second case the invariant is empty. Constructions that lead to Cayley graphs with properties that can be inferred from the Cayley graphs of the building blocks include the direct product, the passage to a subgroup of finite index, and, to a lesser degree, the join $G = \text{gp}(G_1 \cup G_2)$ of two subgroups G_1 and G_2 (see sections A2.3b, B1.2c and B1.2d). In calculating the invariant of groups obtained by these constructions, one fact was used repeatedly, the *independence* of the invariant on the generating system \mathcal{X} (see Theorem A2.3).

Most of the other results obtained in Chapter A rely on Theorem A3.1. It states that for each choice $t \in \mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ with $\chi(t) > 0$, the subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected if, and only if, the following condition holds:

Σ^1 -criterion: for each $y \in \mathcal{Y} \setminus \{t, t^{-1}\}$ there exists a path p_y in $\Gamma(G, \mathcal{X})$ that leads from t to $y \cdot t$ and satisfies the inequality

$$v_\chi(p_y) > v_\chi((1, y)) = \min\{0, \chi(y)\}. \quad (\text{B2.1})$$

This condition, being necessary and sufficient, shows that if a point $[\chi]$ lies in $\Sigma^1(G)$ then an open neighbourhood of $[\chi]$ will be contained in $\Sigma^1(G)$. This fact is a key ingredient in the proofs of the *openness* of Σ^1 (Theorem A3.3) and of the characterization of finitely generated normal subgroups above G' (Theorem A4.1).

The present section focuses on another aspect of the Σ^1 -criterion: it links a geometric condition to the existence of a set of relations with specific properties. Now one may happen to know certain relations among the generators, for instance because the group G is given by a presentation. If so, one can try to select from these relations a subset which fulfills the Σ^1 -criterion for some point in $S(G)$. Each subset for which this strategy is successful yields then a non-empty open subset of $\Sigma^1(G)$, and the union of these sets is a lower bound of the invariant. This simple minded approach yields rarely all of $\Sigma^1(G)$, but it does so in the case of groups defined by a single relator (see Section B4).

The contents of Section B2 are as follows: in B2.1, the Σ^1 -criterion is restated in algebraic terms. As a first application, a lower bound $\psi(\mathcal{R})$ of $\Sigma^1(G)$ is constructed in B2.2; here \mathcal{R} is a set of relators of G . In section B2.3 this lower bound is then worked out for some groups with two generators. In the final part B2.4, an improvement of the lower bound $\psi(\mathcal{R})$ is discussed.

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B2.1 Algebraic version of the Σ^1 -criterion

Let G be a finitely generated group, $\eta: \mathcal{X} \rightarrow G$ a finite system of generators, $\chi: G \rightarrow \mathbb{R}$ a non-zero character and $t \in \mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ an element with $\chi(t) > 0$. The Σ^1 -criterion requires that for each $y \in \mathcal{Y} \setminus \{t, t^{-1}\}$ there be a path p_y from t to $y \cdot t$ which satisfies the inequality $v_\chi(p_y) > v_\chi((1, y))$. This geometric condition will now be restated in algebraic terms.

B2.1a The criterion

Let y be an element of $\mathcal{Y} \setminus \{t, t^{-1}\}$ and let w_y be the word in \mathcal{Y} that describes the path p_y ; so $p_y = (t, w_y)$. Since the path (t, w_y) leads from t to $y \cdot t$, the relations $t \cdot w_y = y \cdot t$ and hence $t^{-1}yt = w_y$ hold in G . Inequality (B2.1) can then be restated by saying that $v_\chi(w_y) > \min\{0, \chi(y)\} - \chi(t) = v_\chi(t^{-1}yt)$.

Consider now the relator $r_y = t^{-1}yt \cdot w_y^{-1}$. One can assume, without loss of generality, that w_y is a freely reduced word. This does not force r_y to be freely reduced, let alone to be cyclically reduced, but the assumption that $v_\chi(t^{-1}yt)$ be smaller than $v_\chi(w_y)$ allows one to find out what can happen. Two cases will be of interest in the sequel. If $\chi(y) = 0$ then $v_\chi(t^{-1}yt) = \chi(t^{-1})$ and so w_y does neither start with t^{-1} nor does it end with t , whence r_y is cyclically reduced. If $\chi(y) > 0$ then $v_\chi(t^{-1}yt) = \chi(t^{-1})$ and so w_y does neither start with t^{-1} nor does it end in yt ; it may, however, end in t . If w_y ends in t , say $w_y = w'_y \cdot t$, then r_y is freely equivalent to the word $t^{-1}y(w'_y)^{-1}$ and this word is cyclically reduced.

The upshot of the preceding analysis is this: if $\chi(y) \geq 0$ and if there exists a word w_y with $v_\chi(w_y) > v_\chi(t^{-1}yt)$ then there exists also a cyclically reduced relator $r'_y = s_1 s_2 \cdots s_k$, starting with $t^{-1}yt$ if $\chi(y) = 0$ and with $t^{-1}y$ if $\chi(y) > 0$, and so that the sequence

$$(\chi(s_1), \chi(s_1 s_2), \dots, \chi(s_1 s_2 \cdots s_{k-1}), \chi(s_1 s_2 \cdots s_{k-1} s_k))$$

assumes its minimum only at $\chi(s_1)$ and $\chi(s_1 s_2)$ in the first case, and only at $\chi(s_1)$ in the second case. We finally pass to the cyclic permutation of the relator r'_y which moves the first letter to the end, and obtain the relator r_y .

The new relators r_y show that the conditions described in equation (B2.2) of the next theorem are necessary.

THEOREM B2.1 *Let $\eta: \mathcal{X} \rightarrow G$ be a finite system of generators, $\chi: G \rightarrow \mathbb{R}$ a non-zero character and $t \in \mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ an element with positive χ -value. Set*

$$\mathcal{Y}_{>0} = \{y \in \mathcal{Y} \mid \chi(y) > 0\} \text{ and } \mathcal{X}_0 = \{x \in \mathcal{X} \mid \chi(x) = 0\}.$$

Then the Σ^1 -criterion holds for \mathcal{Y} , χ and t if, and only if, there exists, for each $y \in (\mathcal{Y}_{>0} \cup \mathcal{X}_0) \setminus \{t\}$ a reduced word $r_y = s_1 s_2 \cdots s_k$ which is a relator of G and has the form

$$r_y = \begin{cases} y \cdots t^{-1} & \text{and } \chi(s_1 \cdots s_i) > 0 \text{ for } 1 < i < k \quad \text{if } \chi(y) > 0, \\ yt \cdots t^{-1} & \text{and } \chi(s_1 \cdots s_i) > 0 \text{ for } 1 < i < k \quad \text{if } \chi(y) = 0. \end{cases} \quad (\text{B2.2})$$

Notes

Proof. The transformations leading to the conditions stated in Theorem B2.1 can be reversed. It follows that there exists, for every $x \in \mathcal{X}$, a sign ε with the following properties: $\chi(x^\varepsilon) \geq 0$ and a path $p_{y,\varepsilon}$ from t to tx^ε can be found so that $v_\chi(p_{y,\varepsilon}) > v_\chi(1, x^\varepsilon)$. But if so, the geometric version of the Σ^1 -criterion holds, for the missing paths can be constructed with the help of Remark A3.2a. \square

In the remainder of section B2, we describe an algorithm that constructs a lower bound $\psi(\mathcal{R})$ of $\Sigma^1(G)$ and then apply it to some examples. The algorithm presupposes that a non-empty set of relators \mathcal{R} be known at the outset; it works with any number of generators. Its outcome, however, is rarely satisfactory if G is generated by more than 2 or 3. There exists an improvement that can cope with a large number of generators; it will be discussed in section B2.4. Its justification will rely on properties of the weaker algorithm.

B2.2 The lower bound $\psi(\mathcal{R})$

We begin by fixing the notation. As usual, G denotes a group, $\eta: \mathcal{X} \rightarrow G$ a finite generating system and \mathcal{Y} the alphabet $\mathcal{X} \cup \mathcal{X}^{-1}$. Set

$$\mathcal{Y}_{>0} = \{y \in \mathcal{Y} \mid \chi(y) > 0\} \text{ and } \mathcal{X}_0 = \{x \in \mathcal{X} \mid \chi(x) = 0\}. \quad (\text{B2.3})$$

Given a set \mathcal{R} of relators of G — in other words, a set of reduced words that belong to the kernel of the epimorphism $\eta_*: F(\mathcal{X}) \twoheadrightarrow G$ — we want to define a subset $\psi(\mathcal{R})$ and then to prove that $\psi(\mathcal{R}) \subseteq \Sigma^1(G)$. In the definition of $\psi(\mathcal{R})$ enter finite sequences of functions f_r , one for each $r \in \mathcal{R}$; they are defined as follows: suppose r has the spelling $s_1 s_2 \cdots s_k$ as a \mathcal{Y} -word and that χ is a non-zero character of G . Then

$$f_r(\chi) = (\chi(s_1), \chi(s_1 s_2), \dots, \chi(s_1 s_2 \cdots s_{k-1}), \chi(s_1 s_2 \cdots s_{k-1} s_k)). \quad (\text{B2.4})$$

The domain of definition of $f_r(\chi)$ will be thought of as being the circle $\mathbb{Z}/k\mathbb{Z}$; in particular, the symbol $f_r(k+1)$ will denote the term $f_r(1)$.

DEFINITION B2.2 Let $t \in \mathcal{Y}$ be an element for which $\mathcal{H}_t = \{[\chi] \in S(G) \mid \chi(t) > 0\}$ is non-empty. The subset $\psi(\mathcal{R})_t$ is made up of the points $[\chi]$ which satisfy the following conditions:

- (i) for every $y \in \mathcal{Y}_{>0} \setminus \{t\}$ there exists a relator $r_y \in \mathcal{R}$ such that the sequence $f_{r_y}(\chi)$ assumes its minimum only once, say in j , and the subword $s_j s_{j+1}$ is either $y^{-1}t$ or $t^{-1}y$;
- (ii) for every $x \in \mathcal{X}_0$ there exists a relator $r_x \in \mathcal{R}$ such that the sequence $f_{r_x}(\chi)$ assumes its minimum in two consecutive indices $j, j+1$, and only there, and that the subword $s_j s_{j+1} s_{j+2}$ is either $t^{-1}xt$ or $t^{-1}x^{-1}t$.

Define $\psi(\mathcal{R})$ to be the union $\bigcup_t \psi(\mathcal{R})_t$ where t ranges over all letters $t \in \mathcal{Y}$ for which $\mathcal{H}_t = \{[\chi] \in S(G) \mid \chi(t) > 0\}$ is non-empty.

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REMARK B2.3 The letters y and t occurring in statement (i) will be called the *letters involved in the minimum*. This minimum can be assumed at the last index k ; if so, $\min f_{r_y} = \chi(r_y) = 0$ and either $r_y = t \cdots y^{-1}$ or $r_y = y \cdots t^{-1}$.

Similarly, we say in the case of statement (ii) that the letters x^ε and t are involved in the minimum if the subword $s_j s_{j+1} s_{j+2}$ equals $t^{-1} x^\varepsilon t$.

EXAMPLE B2.4 The following example illustrates the computation of the set $\psi(\mathcal{R})$ in the case where G_{ab} has rank 1; then $S(G)$ has only 2 points and the determination of $\psi(\mathcal{R})$ is easy.

Let G be given by the presentation $\mathcal{P} = \langle a, b, c \mid r_1, r_2 \rangle$ where

$$r_1 = acacbc a^{-1} b^{-1} c^{-3} \quad \text{and} \quad r_2 = ac^{-1} a^{-1} b^{-2} c^{-1} bcb^2 c. \quad (\text{B2.5})$$

The exponents sums of these relators with respect to the given generators are

$$(\sigma_a(r_1), \sigma_b(r_1), \sigma_c(r_1)) = (1, 0, 0) \quad \text{and} \quad (\sigma_a(r_2), \sigma_b(r_2), \sigma_c(r_2)) = (0, 1, 0).$$

So a and b represent elements in the commutator subgroup of G , whence G_{ab} is an infinite cyclic group generated by the image of c . The points in $S(G)$ are therefore represented by the character

$$\chi: a \mapsto 0, \quad b \mapsto 0, \quad c \mapsto 1$$

and by its opposite $-\chi$. The functions f_{r_1} and f_{r_2} are easily computed:

$$\begin{aligned} f_{r_1}(\chi) &= (0, 1, 1, 2, 2, 3, 3, 3, 0), \\ f_{r_2}(\chi) &= (0, -1, -1, -1, -1, -2, -2, -1, -1, 0), \\ f_{r_1}(-\chi) &= (0, -1, -1, -2, -2, -3, -3, -3, 0), \\ f_{r_2}(-\chi) &= (0, 1, 1, 1, 1, 2, 2, 1, 1, 0). \end{aligned}$$

In the case of the character χ , both sequences assume their minima exactly twice and at consecutive indices; in the case of f_{r_1} the minima occur at the last letter c^{-1} and the first letter a ; in the case of f_{r_2} the minima occur at the fifth letter c^{-1} and the sixth letter b . The letter c can play the rôle of t — it is actually the only choice for t ; it is involved in both minima, while a is involved in the minimum of f_{r_1} and b is involved in f_{r_2} . The point $[\chi]$ thus lies in $\psi(\mathcal{R})$.

Now to the character $-\chi$. The sequence $f_{r_1}(-\chi)$ assumes its minimum thrice; as there are only two relators in \mathcal{R} the point $[-\chi]$ is therefore outside $\psi(\mathcal{R})$. So $\psi(\mathcal{R}) = \{[\chi]\}$.

The next result explains our interest in $\psi(\mathcal{R})$:

PROPOSITION B2.5 *Let $\eta: \mathcal{X} \rightarrow G$ be a finite generating system of the group G and let \mathcal{R} be a non-empty set of cyclically reduced words in $\mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$. If \mathcal{R} is made up of relators of G then $\psi(\mathcal{R})$ is an open subset of $\Sigma^1(G)$.*

Notes

Proof. The openness of $\psi(\mathcal{R})$ follows directly from the continuity of the function $\chi \mapsto v_r(\chi)$ and the definition of $\psi(\mathcal{R})$.

Suppose $[\chi] \in \psi(\mathcal{R})$. By the definition of $\psi(\mathcal{R})$ there exist then a letter $t \in \mathcal{Y}_{>0}$, a relator r_y for each $y \in \mathcal{Y}_{>0} \setminus \{t\}$ and a relator r_x for each $x \in \mathcal{X}_0$, all in such a way that the properties enunciated in statements (i) and (ii) hold. We want to show that suitable cyclic permutations of one of the words r_y and r_y^{-1} , and of one of the words r_x and r_x^{-1} satisfy condition (B2.2).

Suppose first $y \in \mathcal{Y}_{>0}$. Then there exists a relator $r_y = s_1 \cdots s_k \in \mathcal{R}$ such that the sequence

$$f_{r_y}(\chi) = (\chi(s_1), \chi(s_1 s_2), \dots, \chi(s_1 s_2 \cdots s_j), \dots, \chi(r_y))$$

has a unique minimum and that the letters t and y are involved in this minimum. If the minimum occurs in k , the relator has either the form $t \cdots y^{-1}$ or $y \cdots t^{-1}$ and all its proper initial segments u have positive χ -value. If the relator starts with y it satisfies therefore the requirement stated in condition (B2.2); if it starts with t , its inverse will be a relator satisfying the condition.

If the minimum occurs in $j < k$, we set $u = s_1 \cdots s_j$ and $w = s_{j+1} \cdots s_k$ and consider the cyclic permutation $w \cdot u$ of r_y . We claim that all the values

$$\chi(s_{j+1}), \chi(s_{j+1} s_2), \dots, \chi(w), \chi(w \cdot s_1), \chi(w \cdot s_1 s_2), \dots, \chi(w \cdot s_1 \cdots s_{j-1})$$

are positive. To see this, notice that $\chi(u)$ is the minimum of the sequence $f_{r_y}(\chi)$. It follows firstly that $\chi(u \cdot s_{j+1} \cdots s_\ell) > \chi(u)$, whence $\chi(s_{j+1} \cdots s_\ell) > 0$ for $\ell = j+1, j+2, \dots, k$; and then that $\chi(s_1 \cdots s_h) > \chi(u)$, whence

$$\chi(w \cdot s_1 \cdots s_h) = -\chi(u) + \chi(s_1 \cdots s_h) > 0.$$

Recall now that the letters t and y are involved in the minimum of the sequence $f_{r_y}(\chi)$. It follows that the cyclic permutation $w \cdot u$ has either the form $t \cdots y^{-1}$ or $y \cdots t^{-1}$. In the first case the word $u \cdot w$ itself has properties required by condition (B2.2); in the second case, the relator $(w \cdot u)^{-1}$ will have them.

Let's now investigate the case of a letter $x \in \mathcal{X}_0$. By the definition of $\psi(\mathcal{R})$ there exists a relator $r_x = s_1 \cdots s_k$ such that the minimum of the sequence

$$f_{r_x}(\chi) = (\chi(s_1), \chi(s_1 s_2), \dots, \chi(s_1 s_2 \cdots s_j), \dots, \chi(r_x))$$

occurs twice, at two consecutive indices $j, j+1$, and so that $s_j s_{j+1} s_{j+2} = t^{-1} x^\varepsilon t$ for some sign ε . Consider then the word r obtained from r_x by cyclic permutation which starts with the letter s_{j+1} . This word is a relator of the form $x^\varepsilon t \cdots t^{-1}$; one sees as in the first part of the verification that χ assumes positive values on all its proper initial segments. If $\varepsilon = 1$, this relator satisfies therefore condition (B2.2) for a letter y with $y \in \mathcal{X}_0$. If $\varepsilon = -1$ the cyclically permuted relator has the form $x^{-1} t \cdots t^{-1}$. Its inverse is then of the form $t \cdots t^{-1} x$ and has positive χ -values on all proper initial segments. By moving the last letter to the front, one arrives at a relator satisfying condition (B2.2). \square

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B2.2a Geometrical reformulation

If the rank of the abelianized group G_{ab} is larger than 1, the sphere $S(G)$ has infinitely many points and $\psi(\mathcal{R})$ cannot be determined by investigating the points $[\chi]$ one after another. In some cases, a geometric reformulation will then help. It splits the evaluation of the sequences into two parts: one uses first the relators to produce sequences with values in a Euclidean lattice \mathbb{Z}^k ; in the second part, these sequences are composed with the linear forms $x \mapsto \langle u, x \rangle$; here u ranges over the unit sphere of \mathbb{E}^k . The details are as follows.

Let $\vartheta: G \twoheadrightarrow G_{\text{ab}} \rightarrow \mathbb{Z}^k$ be an epimorphism of G onto the standard lattice in \mathbb{R}^k equipped with the usual scalar product. Then ϑ induces an embedding

$$\sigma(\vartheta): \mathbb{S}^{k-1} \xrightarrow{\sim} S(G, \ker \vartheta) \hookrightarrow S(G), \quad u \mapsto [g \mapsto \langle u, \vartheta(g) \rangle] \quad (\text{B2.6})$$

of spheres (see section A1.1d). Every relator $r = s_1 s_2 \cdots c_k$ of G gives rise to a sequence

$$f_r(\vartheta) = (\vartheta(s_1), \vartheta(s_1) + \vartheta(s_2), \dots, \vartheta(s_1) + \cdots + \vartheta(s_k)) \quad (\text{B2.7})$$

of lattice points. The points of this sequence are the vertices of a path \bar{p} in the Cayley graph $\Gamma(\mathbb{Z}^k, \mathcal{X})$ that ends in the origin; we will think of \bar{p} as being a loop, starting at and ending in the origin. If $k = 2$, this loop can easily be depicted. As the examples in the next section will reveal, such a visualization can be helpful in determining $\psi(\mathcal{R})$.

B2.3 Application to groups with two generators

The next aim is the computation of $\psi(\mathcal{R})$ in the simplest interesting set-up, in that of a group with 2 generators. The definition of $\psi(\mathcal{R})$ simplifies then considerably.

Suppose G is generated a and b , and \mathcal{R} is a set of cyclically reduced words in a, a^{-1} and b, b^{-1} made up of relators of G . Given a non-zero character χ of G , two cases arise: if $\chi(a)$ and $\chi(b)$ are both non-zero, then $\mathcal{Y} = \{a, a^{-1}, b, b^{-1}\}$ contains two elements, say y_1, y_2 , with positive values; if now $r \in \mathcal{R}$ is a relator such that the associated sequence $f_r(\chi)$ has a unique minimum, then y_1, y_2 must be the letters involved in this minimum. Statement (i) in the definition B2.2 of $\psi(\mathcal{R})$ is then satisfied for $t = y_1$ as well as $t = y_2$.

Suppose now that $\chi(a) = 0$. Then there exists a sign ε so that $\chi(b^\varepsilon)$ is positive and b^ε is the only element in \mathcal{Y} that is eligible for t . In this situation, we seek a relator $r \in \mathcal{R}$ whose associated sequence assumes its minimum at consecutive indices, and only there. The letters involved in this minimum are the automatically t and one of a, a^{-1} . The case where $\chi(b) = 0$ is similar.

Definition B2.2 can therefore be restated as follows:

DEFINITION B2.6 Let G be a group generated by two elements a, b and let \mathcal{R} be a set of cyclically reduced words in a, a^{-1}, b and b^{-1} which are relators of G . Then a point $[\chi] \in S(G)$ belongs to $\psi(\mathcal{R})$ if, and only if, the following condition holds:

Notes

if $\chi(a_1)$ and $\chi(a_2)$ are both non-zero, the minimum of some sequence $f_r(\chi)$ with $r \in \mathcal{R}$ occurs only once; if one of $\chi(a)$, $\chi(b)$ is zero, the minimum of some sequence $f_r(\chi)$ with $r \in \mathcal{R}$ occurs at two consecutive indices and only there.

In the remainder of this section B2.3, the above proposition will be used in the geometrical interpretation described in B2.2a.

B2.3a Groups with a single defining relator

Let $\langle a, b \mid r \rangle$ be a presentation with a single defining relator r which has exponent sum 0 in each of the generators, say

$$r = a^{-1}b^{-1}ab^2a^{-1}b^{-1}a^2b^{-1}a^{-1}ba^{-1}bab^{-1}. \quad (\text{B2.8})$$

The assignments $a \mapsto (1, 0)$ and $b \mapsto (0, 1)$ extend to an epimorphism $\vartheta: G \rightarrow \mathbb{Z}^2$. Under this epimorphism, the relator gives rise to the loop $\bar{p} = ((0, 0), r)$ in the Cayley graph $\bar{\Gamma} = \Gamma(\mathbb{Z}^2, \{a, b\})$; it is indicated on the left of Figure B.6.

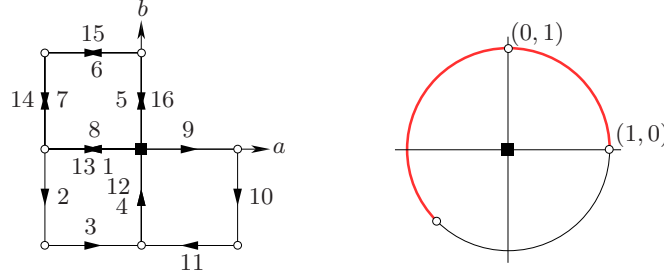


Figure B.6: The loop \bar{p} in $\Gamma(\mathbb{Z}^2, \{a, b\})$ and the subset $\sigma(\theta)^{-1}(\psi(\mathcal{R}))$

A word on the interpretation of this figure is in order. The path \bar{p} is not simple; worse, it contains not only multiple vertices, but edges traversed more than once. One way of dealing with this problem is to set off multiple occurrences of the same edge from one another (see, e. g., [Bro87b, p. 493]). In Figure B.6 a different rendering is chosen: the edges are numbered consecutively, with the position number written on the left of the oriented edge. The path starts at the origin, indicated by the black square in the middle of the figure, then goes around the left, lower square in counterclockwise fashion, then counterclockwise around the left, upper square, then clockwise around the right square and finally in clockwise fashion around the upper left square.

The determination of the subset $\psi(\{r\})$ can now be carried out like this. We seek unit vectors $u = (u_1, u_2) \in \mathbb{S}^1$ for which the function $h_u: x \mapsto \langle u, x \rangle$ assumes its minimum at most twice along the vertices of the path \bar{p} , the vertices being counted with multiplicity. This condition is fulfilled if $u_1 > 0$ and $u_2 > 0$; then the minimum is assumed in $(-1, -1)$ and only there. In view of definition B2.6 the

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image of the open first quadrant of \mathbb{S}^1 under $\sigma(\vartheta): \mathbb{S}^1 \xrightarrow{\sim} S(G)$ belongs therefore to $\psi(\{r\})$. If $u_1 < 0 < u_2$, the function h_u assumes its minimum in $(1, -1)$ and only there, whence the image of the open second quadrant of \mathbb{S}^1 lies in $\psi(\{r\})$.

If $u = (-1, 0)$, the function f_u assumes its minimum at the endpoints of the tenth edge of the path \bar{p} and only there; the second part of the condition stated in definition B2.6 therefore applies and shows that $[\chi_u]$ lies in $\psi(\{r\})$. If, finally, $u_1 < u_2 < 0$, the minimum is assumed in $(1, 0)$, and only there; so $\psi(\{r\})$ contains a further arc. The union of the four subsets detected constitute all of $\psi(\{r\})$; its preimage is depicted in red on the right of Figure B.6.

REMARK B2.7 The subset $\psi(\{r\})$ coincides with $\Sigma^1(G)$; see Theorem B4.1.

B2.3b Groups of PL-homeomorphisms

The second example presents an instance where the invariant can be determined completely, in spite of the fact that only some relators, but not a presentation, are known. The example fits into a larger class of groups (see [BNS87, Section 8]).

Let f and g be piecewise linear homeomorphisms of the unit interval $[0, 1]$ that satisfy the following requirements for some positive real numbers $\varepsilon < \delta$:

$$f(t) < t \text{ for } t \in]0, 1[, \quad (\text{B2.9})$$

$$f(t) = g(t) \text{ for } t \leq \varepsilon, \quad (\text{B2.10})$$

$$g(t) = t \text{ for } t \geq \delta. \quad (\text{B2.11})$$

Let G denote the group generated by these homeomorphisms f and g .

Requirements (B2.10) and (B2.11) imply that G_{ab} is free-Abelian of rank 2. Indeed, the derivative in 0 induces a homomorphism of G into the multiplicative group of $\mathbb{R}_{>0}$; upon composing it with the natural logarithm \ln , one obtains a character $\chi_l: G \rightarrow \mathbb{R}$; similarly, the composition of the derivative in 1 with \ln yields a character χ_r of G . The claim now follows from the facts that

$$\chi_l(f \circ g^{-1}) = 0, \quad \chi_l(g^{-1}) > 0, \quad \text{and} \quad \chi_r(f \circ g^{-1}) > 0, \quad \chi_r(g^{-1}) = 0. \quad (\text{B2.12})$$

The characters χ_l and χ_r enter into the description of $\Sigma^1(G)$ as follows:

PROPOSITION B2.8 *Let G denote the group generated by two PL-homeomorphisms f, g of the unit interval $[0, 1]$ satisfying requirements (B2.9) through (B2.11) for some real numbers $\varepsilon < \delta < 1$. Then G_{ab} is free Abelian of rank 2 and*

$$\Sigma^1(G) = S(G) \setminus \{[\chi_l], [\chi_r]\}. \quad (\text{B2.13})$$

The proof divides into two parts of different sorts. To show that the points $[\chi_l]$ and $[\chi_r]$ lie outside $\Sigma^1(G)$, one analyses the kernels of χ_l, χ_r and then employs Proposition B2.9, a result on descending HNN-extensions. To establish that all other points of $S(G)$ belong to $\Sigma^1(G)$ one determines some relators r_m satisfied by the generators f, g and calculates $\psi(\{r_m\})$. An appeal to proposition B2.5 and a short auxiliary computation then yield the desired conclusion.

Notes

Proof of proposition B2.8: part 1. We first show that $[\chi_r] \notin \Sigma^1(G)$. To do so, we introduce the *support* of a permutation $h \in G$; it is defined to be the subset $\{x \in X \mid h(x) \neq x\}$. If h_1 and h_2 are permutations, the formula

$$\text{supp}(h_1 \circ h_2 \circ h_1^{-1}) = h_1(\text{supp}(h_2)). \quad (\text{B2.14})$$

holds, as one verifies easily. Let now δ_0 be the supremum of $\text{supp}(g)$; assumptions (B2.9) through B2.11 imply that $\varepsilon < \delta_0 \leq \delta < 1$. The subset

$$H = \{h \in G \mid h(t) = t \text{ for } t \geq \delta_0\}.$$

is actually a subgroup of G and it satisfies the inclusion

$$f \circ H \circ f^{-1} = \{h \in G \mid h(t) = t \text{ for } t \geq f(\delta_0)\} \subsetneq H$$

for $g \in H \setminus f \circ H \circ f^{-1}$. It shows that G is a *properly descending* HNN-extension with base group H and stable letter f . Proposition B2.9 allows us therefore to conclude that $[\chi_r] \notin \Sigma^1(G)$. One can verify similarly that $[\chi_l] \notin \Sigma^1(G)$.

The following proposition provides a tool for showing that a rank 1 point lies *outside* Σ^1 . It can be generalized to points of arbitrary rank (see Proposition C1.15).

PROPOSITION B2.9 *Let $\chi: G \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ be a rank 1 character of G and let $t \in G$ be an element with $\chi(t) = 1$. If G is a strictly descending HNN-extension $\langle H, t \mid tHt^{-1} \subsetneq H \rangle$ over a subgroup H of $\ker \chi$ then $[\chi] \notin \Sigma^1(G)$.*

Proof. Set $N = \ker \chi$ and consider a finitely generated subgroup $B \subset N$ with $B \subset tBt^{-1}$. Since B is finitely generated and H is the basis of a descending HNN-extension with stable letter t , there exists a positive integer ℓ_0 such that $B \subset t^{-\ell}Ht^\ell$. For all $j \geq 0$ the following chain of inclusions

$$t^j B t^{-j} \subseteq t^j (t^{-\ell} H t^\ell) t^{-j} = t^{-\ell} (t^j H t^{-j}) t^\ell \subseteq t^{-\ell} H t^\ell$$

is then valid. Since $t^{-\ell} H t^\ell \subsetneq N$ this chain precludes the sequence $j \mapsto t^j B t^{-j}$ from sweeping out N , whence $[\chi] \notin \Sigma^1(G)$ by Proposition A3.4. \square

Proof of Proposition B2.8: part 2. Requirements (B2.9) through (B2.11) imply some distinctive relations among the generators f and g . To detect them, we proceed as follows. Requirement (B2.9) implies, first of all that there exists a positive exponent m_0 with $f^{m_0}(\delta) \leq \varepsilon$. Next requirement (B2.11) and formula (B2.14) give rise to the chain of inclusions

$$\text{supp}(f^m \circ g \circ f^{-m}) = f^m(\text{supp}(g)) \subseteq f^m([0, \delta]) \subseteq [0, \varepsilon].$$

Set $g_m = f^m \circ g \circ f^{-m}$. Then the commutativity relation

$$(g^{-1} \circ f) \circ g_m = g_m \circ (g^{-1} \circ f) \quad (\text{B2.15})$$

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is valid for every $m \geq m_0$; indeed, the functions f and g agree on the interval $[0, \varepsilon]$, whence the support of $g^{-1} \circ f$ is contained in $[\varepsilon, 1]$ and so disjoint from the support of g_m . Relation (B2.15) is equivalent to the relator

$$r_m = f \circ (f^m \circ g \circ f^{-m}) \circ f^{-1} \circ g \circ (f^m \circ g^{-1} \circ f^{-m}) \circ g^{-1}. \quad (\text{B2.16})$$

The following Lemma then allows us to conclude that $\Sigma^1(G) \supseteq S(G) \setminus \{[\chi_l], [\chi_r]\}$.

LEMMA B2.10 *Given a positive integer m , set*

$$r_m = a^{m+1} \cdot b \cdot a^{-(m+1)} \cdot b \cdot a^m \cdot b^{-1} \cdot a^{-m} \cdot b^{-1} \quad (\text{B2.17})$$

and define G_m to be the group given by the presentation $\langle a, b \mid r_m, r_{m+1} \rangle$. Then G_{ab} is free-Abelian of rank 2 and $\Sigma^1(G) \supseteq S(G) \setminus \{[\chi_1], [\chi_2]\}$ with χ_1, χ_2 the characters given by

$$\chi_1(a) = 1, \quad \chi_1(b) = 0 \quad \text{and} \quad \chi_2(a) = \chi_2(b) = -1. \quad (\text{B2.18})$$

Proof. Let $\vartheta: G \rightarrow \mathbb{Z}^2$ be the epimorphism sending a to $(1, 0)$ and b to $(0, 1)$. Under this epimorphism the relator r_m give rise to a closed path $\bar{p}_m = ((0, 0), r_m)$ in the Cayley graph $\Gamma(\mathbb{Z}^2, \{a, b\})$; for $m > 1$ it has the form indicated on the left of Figure B.7. The corresponding subset $\psi(\{r_m\})$ is depicted on the right of the figure. Here χ_3 and χ_4 denote the characters

$$\chi_3(a) = 0, \quad \chi_3(b) = 1 \quad \text{and} \quad \chi_4 = -\chi_3.$$

If $m = 1$, the subset $\psi(\{r_1\})$ contains also the point $[\chi_4]$.

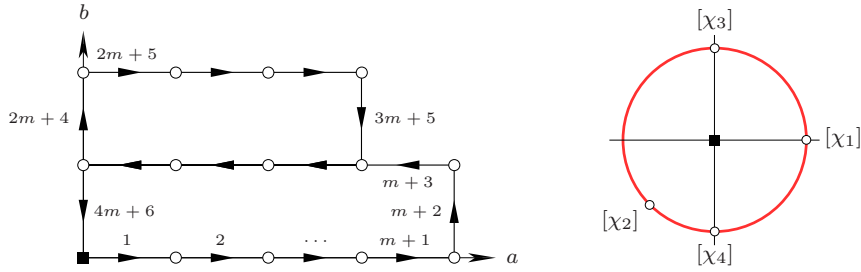


Figure B.7: Path \bar{p}_m in $\Gamma(\mathbb{Z}^2, \{a, b\})$ and subset $\psi(\{r_m\}) \subset S(Q)$

We are left with proving that $[\chi_3]$ and $[\chi_4]$ lie in $\Sigma^1(G_m)$; to this end we derive some auxiliary relations from the relators r_m and r_{m+1} . These relators are equivalent to the relations

$$a^{m+1}b = ba^mb \quad \text{and} \quad a^{m+2}b = ba^{m+1}b. \quad (\text{B2.19})$$

Notes

If the first relation is conjugated by a , the left members of both relations agree; the right members then lead to the relator

$$r_3 = aba^m ba^{-m} b^{-1} a^{-1} \cdot ba^{m+1} b^{-1} a^{-m-1} b^{-1}$$

The two relations allow one also to derive the following chain of transformations

$$a^{m+1} b = ba^m b = ba^{-1} b^{-1} (ba^{m+1} b) = ba^{-1} b^{-1} a^{m+2} b,$$

which is equivalent to the relator

$$r_4 = a^{m+1} ba^{-m-1} \cdot ba^{-1} b^{-1} a^{m+2} b^{-1} a^{-m-2} bab^{-1}.$$

Consider now the sequences $f_{r_3}(\chi_3)$ and $f_{r_4}(\chi_4)$. The first of them assumes its minimum twice, at the first letter a and at r_3 ; by definition B2.6 and Proposition B2.5, the point $[\chi_3]$ lies therefore in $\Sigma^1(G_m)$. The second sequence assume its minimum again twice, at the initial segment $a^{m+1} ba^{-m-1} b$ and at the following one, whence $[\chi_4] \in \Sigma^1(G_m)$ by the same reasons. \square

REMARKS B2.11 a) The calculations in the previous example can be summarized as follows: let G be the group generated by two piecewise linear homeomorphisms f and g of the unit interval $[0, 1]$. Suppose the generators satisfy the requirements (B2.9) through (B2.11). Then the complement $\Sigma^1(G)^c$ of the invariant consists of only two points, represented by the characters $\ln \circ D_0$ and $\ln \circ D_1$. Here D_0 denotes the derivative at the fixed point 0 and D_1 denotes the derivative at the left global fixed point 1 of the interval $[0, 1]$.

This result shows that the invariant does not depend on the details of the functions f and g . By [BNS87, Thm. 8.1] the same conclusion holds for a far larger class of groups made up of piecewise linear homeomorphisms.

b) The abelianization G_{ab} of G is free abelian of rank 2. The group G contains therefore infinitely many normal subgroups N with infinite cyclic quotient. All of them are finitely generated, except the kernels of the derivatives with respect to the end points 0 or 1 (use the previous calculation of $\Sigma^1(G)$ and Corollary A4.3).

B2.4 The lower bound $\Psi(\mathcal{R})$

In this section, an improved lower bound for $\Sigma^1(G)$ is derived. To explain how it differs from $\psi(\mathcal{R})$, we recall the strategy underlying the construction of $\psi(\mathcal{R})$.

The algebraic Σ^1 -criterion says that a non-zero-character χ represents a point of $\Sigma^1(G)$ if $\text{card}(\mathcal{X}) - 1$ relators with specific properties can be found. In the case of $\psi(\mathcal{R})$, these relators are sought in the symmetrization $\text{sym}(\mathcal{R})$ of \mathcal{R} , that is the set consisting of the cyclic permutations of the elements of $\mathcal{R} \cup \mathcal{R}^{-1}$. In the improvement, one looks for relators in a larger set consisting of products whose factors are either elements of $\text{sym}(\mathcal{R})$ or conjugates xrx^{-1} with $x \in \mathcal{X}$ and $r \in \text{sym}(\mathcal{R})$.

B2.4a Defining $\Psi(\mathcal{R})$

We begin by fixing the notation. As before, $\eta: \mathcal{X} \rightarrow G$ denotes a finite system of generators of the group G and $\mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ the associated set of letters. Next, \mathcal{R} is a set of cyclically reduced relators in \mathcal{Y} . Given $r \in \mathcal{R}$ and a non-zero character $\chi: G \rightarrow \mathbb{R}$, let $f_r(\chi)$ denote the sequence

$$(\chi(s_1), \chi(s_1 s_2), \dots, \chi(s_1 s_2 \cdots s_{k-1}), \chi(s_1 s_2 \cdots s_{k-1} s_k)) \quad (\text{B2.20})$$

and view its domain as a circle with k adjacent to 1. Finally, set

$$\mathcal{Y}_{>0} = \{y \in \mathcal{Y} \mid \chi(y) > 0\} \text{ and } \mathcal{X}_0 = \{x \in \mathcal{X} \mid \chi(x) = 0\}.$$

We next introduce a graph that will be used in the definition of the subset $\Psi(\mathcal{R})$ of $S(G)$.

DEFINITION B2.12 Given a non-zero character χ of G , put

$$\begin{aligned} \mathcal{R}_{\chi,+} &= \{r \in \mathcal{R} \mid f_r(\chi) \text{ assumes its minimum once}\}, \\ \mathcal{R}_{\chi,0} &= \{r \in \mathcal{R} \mid f_r(\chi) \text{ assumes its minimum twice, at adjacent indices}\}. \end{aligned}$$

Using $\mathcal{R}_{\chi,+}$, define a graph $\mathcal{G}_{\mathcal{R}}(\chi)$ with vertex set $\mathcal{Y}_{>0}$ and edge set

$$\{\{y_1, y_2\} \mid y_1 \text{ and } y_2 \text{ are involved in } \min f_r(\chi) \text{ for some } r \in \mathcal{R}_{\chi,+}\}. \quad (\text{B2.21})$$

DEFINITION B2.13 A point $[\chi] \in S(G)$ belongs to $\Psi(\mathcal{R})$ if, and only if, the following conditions are satisfied:

- (i) the graph $\mathcal{G}_{\mathcal{R}}(\chi)$ is connected;
- (ii) for every generator $x \in \mathcal{X}_0$ there exist a relator $r = s_1 \cdots s_k \in \mathcal{R}_{\chi,0}$ so that $f_r(\chi)$ assumes its minimum at two consecutive indices $j, j+1$ and the subword $s_j s_{j+1} s_{j+2}$ has the form $y_1^{-1} x^\varepsilon y_2$ with $y_1, y_2 \in \mathcal{Y}_{>0}$ and $\varepsilon \in \{1, -1\}$.

We shall later show that $\Psi(\mathcal{R})$ is a subset of $\Sigma^1(G)$ and that it contains $\psi(\mathcal{R})$. But first we give an example which illustrates the definition of $\Psi(\mathcal{R})$.

EXAMPLE B2.14 Let G be given by the presentation $\langle a, b, c \mid r_1, r_2 \rangle$ with

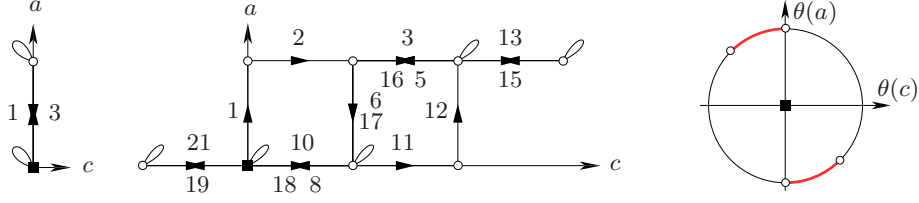
$$r_1 = abAB \quad \text{and} \quad r_2 = ac^2bCABcb^2acBC^2AC^2Bc. \quad (\text{B2.22})$$

Here A, B are short for a^{-1} and b^{-1} . The exponent sums of the second relator are

$$(\sigma_a(r_2), \sigma_b(r_2), \sigma_c(r_2)) = (0, 1, 0).$$

It follows that G_{ab} is free-Abelian of rank 2 and that G admits an epimorphism $\vartheta: G \rightarrow \mathbb{Z}^2$ taking a to $(0, 1)$ and c to $(1, 0)$ (and, of course, b to $(0, 0)$).²

Notes

Figure B.8: Loops of the relators r_1 and r_2 and the set Ψ'

The paths $\bar{p}_{r_1}, \bar{p}_{r_2}$ in the Cayley graph $\Gamma = (\mathbb{Z}^2, \{a, b, c\})$ have edges which are loops; in Figure B.8 they are indicated by small leaves. The straight edges are numbered as usual; the leaves are not labelled, but they are taken into account in the numbering of the straight edges.

Let us determine $\Psi' = \sigma(\theta)^{-1}(\Psi(\mathcal{R}))$ for $\mathcal{R} = \{r_1, r_2\}$. Two cases arise. If $\chi_u(a)$ and $\chi_u(c)$ are both non-zero, the graph $\mathcal{G}_{\mathcal{R}}(\chi)$ has two vertices and so it is only connected if one of the graphs in Figure B.8 contributes an edge to it. Only the path p_{r_2} can do that, and only if the supporting half plane \mathcal{H}_u touches the convex hull in a single vertex with no leaf attached to it. (Recall that the unit vector u points into the half plane \mathcal{H}_u .) These conditions are fulfilled if, and only if, u lies in the first part of the second or the fourth open quadrant. If the stated conditions are fulfilled, the relator r_1 shows that b satisfies condition (ii) in definition B2.13. It follows that $\Psi(\mathcal{R})$ contains the described arcs.

If one of $\chi_u(a)$ or $\chi_u(c)$ is zero, the graph $\mathcal{G}_{\mathcal{R}}(\chi)$ is connected and condition (ii) holds only if each of the sequences $f_{r_1}(\chi)$ and $f_{r_2}(\chi)$ assumes its minimum precisely twice, at consecutive indices. The path p_{r_2} shows that $f_{r_2}(\chi)$ assumes its minimum only twice if $\chi(a) = 0$. As the first sequence $f_{r_1}(\chi)$ is then constant, $\Psi(\mathcal{R})$ contains none of the 4 points under discussion, and so $\Psi(\mathcal{R})$ is the union of the two arcs found before.

B2.4b Verification that $\Psi(\mathcal{R})$ is a lower bound

The following result is a variant of [GM98, Theorem A], established by Susan Garner Garille and John Meier. It reveals that $\Psi(\mathcal{R})$ is a lower bound of the invariant and that this new bound improves on $\psi(\mathcal{R})$.

PROPOSITION B2.15 *Let $\eta: \mathcal{X} \rightarrow G$ be a finite generating system of the group G and \mathcal{R} a non-empty set of cyclically reduced words in $\mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$. If \mathcal{R} is made up of relators of G then $\Psi(\mathcal{R})$ is an open subset of $S(G)$ and $\psi(\mathcal{R}) \subseteq \Psi(\mathcal{R}) \subseteq \Sigma^1(G)$.*

Proof. The openness of $\Psi(\mathcal{R})$ follows directly from the continuity of the function $\chi \mapsto v_r(\chi)$ and the definition of $\Psi(\mathcal{R})$.

²This choice reflects the disposition of the axes in Figure B.8.

First we show that $\psi(\mathcal{R}) \subseteq \Psi(\mathcal{R})$. Suppose χ represents a point of $\psi(\mathcal{R})$. By definition B2.2 there exist then a letter $t \in \mathcal{Y}_{>0}$ and, for each $y \in \mathcal{Y}_{>0} \setminus \{t\}$, a relator $r_y \in \mathcal{R}_{\chi,+}$ such that t and y are involved in the unique minimum of $f_r(\chi)$. The graph $\mathcal{G}_{\mathcal{R}}(\chi)$ thus contains for every vertex y distinct from t an edge connecting y and t and so it is connected. Moreover, there exists by (ii) of definition B2.2, for every $x \in \mathcal{X}_0$, a relator $r_x \in \mathcal{R}_{\chi,0}$ such that the minimum of $f_{(r_x,\chi)}$ occurs at j and $j+1$, say, and so that the subword $s_j s_{j+1} s_{j+2}$ equals $t^{-1} x^\varepsilon t$ with $|\varepsilon| = 1$. These relators satisfy then requirement (ii) of the definition of $\Psi(\mathcal{R})$, and so $[\chi] \in \Psi(\mathcal{R})$.

We verify next that $\Psi(\mathcal{R}) \subseteq \Sigma^1(G)$. Suppose $[\chi]$ is in $\Psi(\mathcal{R})$. By condition (i) in definition B2.13 the graph $\mathcal{G}_{\mathcal{R}}(\chi)$ is then connected. Given an edge $\{y_1, y_2\}$ of this graph there exists a relator $r \in \mathcal{R}$ such that the sequence $f_r(\chi)$ has a unique minimum, say at j , and that y_1 and y_2 are involved in this minimum. So r is a word of the form $w_1 y_1^{-1} y_2 w_2$ or the form $w_1 y_2^{-1} y_1 w_2$; there is no loss in assuming that we are in first case. A suitable cyclic permutation r' of r is then equal to $y_2 w_2 w_1 y_1^{-1}$ and the χ -values of all proper initial segments of r' are positive. Notice that the inverse of r' has the form $y_1 \cdots y_2^{-1}$ and satisfies the condition that all its proper initial segment have positive χ -values. The vertex set of the graph $\mathcal{G}_{\mathcal{R}}(\chi)$ is not empty for χ is not the zero-homomorphism; there exists therefore a vertex in the graph; call it t .

Consider next a vertex $y \in \mathcal{Y}_{>0} \setminus \{t\}$. Since the graph $\mathcal{G}_{\mathcal{R}}(\chi)$ is connected; there is an edge path from y to t , say $p = (y = y_1, y_2, \dots, y_k = t)$. By the previous paragraph there exists, for each $i = 1, \dots, k-1$, a relator r'_i which has the form $r'_i = y_i w_i y_{i+1}^{-1}$ and enjoys the property that all its proper initial segments have positive χ -values. The product

$$(y w_1 y_2^{-1}) \cdot (y_2 w_2 y_3^{-1}) \cdots (y_{k-2} w_{k-1} y_{k-1}^{-1}) \cdot (y_{k-1} w_k t^{-1})$$

is then a word that starts with y , ends with t^{-1} and is a relator of G . It simplifies to $r'(y \rightarrow t) = y w_1 w_2 \cdots w_k t^{-1}$; this word $r(y \mapsto t)$ may not be reduced, but it has the property that all its proper initial segments have positive χ -values. The unique freely reduced word r_y that is freely equivalent to $r'(y \mapsto t)$ starts then with y , ends with t^{-1} and has positive χ -value on all of its proper initial segments. It satisfies thus the requirements of the relator denoted r_y in equation (B2.2).

Consider, finally, $x \in \mathcal{X}_0$. By condition (ii) in definition B2.13, there exists then a relator $r_x \in \mathcal{R}$ with the following properties: the sequence $f_r(\chi)$ attains its minimum exactly twice, at consecutive indices $j, j+1$, say, and the subword $s_j s_{j+1} s_{j+2}$ has the form $y_1^{-1} x^\varepsilon y_2$ with $y_1, y_2 \in \mathcal{Y}_{>0}$ and $\varepsilon \in \{1, -1\}$. If $\varepsilon = 1$, a cyclic permutation of r has the form $x y_2 \cdots y_1^{-1}$ and χ is positive on each proper initial segment, except, of course, on the first one x ; if $\varepsilon = -1$, a cyclic permutation r^{-1} will have the form $x y_1 \cdots y_2^{-1}$ and enjoy the previously stated property.

So far we know that there exists, for every $x \in \mathcal{X}_0$, a reduced relator r_x which has the form $x y_1 w y_2^{-1}$ and positive χ -values on each proper initial segment except the first one. Various cases now arise. If $y_1 = y_2 = t$, the relator r_x itself has the form stated in (B2.2). If $y_2 \neq t$, there exists, by the first part of the proof, a reduced relator $r(y_2 \rightarrow t)$ having the form $y_2 w_2 t^{-1}$ and positive χ -values on all of

its proper initial segments. The product

$$r_x \cdot r_{y_2 \rightarrow t} = xy_1w_1y_2^{-1} \cdot y_2w_2t^{-1}$$

is then a relator that simplifies to the relator $xy_1w_1w_2t^{-1}$ and has positive χ -values on its proper initial segments distinct from the last one. If $y_1 = t$, this relator fulfills requirement (B2.2). Otherwise, there exists a relator $r(y_1 \rightarrow t) = y_1wt^{-1}$ with positive χ -values on its proper initial segments. The product

$$xr(y_1 \rightarrow t)^{-1}x^{-1} \cdot (xy_1w_1w_2y_2^{-1})$$

simplifies then to a relator of the form xtw_4t^{-1} that satisfies requirement (B2.2).

All taken together we have shown that there exists, for every $y \in \mathcal{Y}_{>0}$, a reduced word of the form $ys_2 \cdots t^{-1}$ and, for every $x \in \mathcal{X}_0$, a reduced word of the form $xts_3 \cdots t^{-1}$, all in such a way that each of these words is a relator of G for which condition (B2.2) holds. Theorem B2.1 thus implies that $[\chi] \in \Sigma^1(G)$. \square

B2.4c Application to graph groups

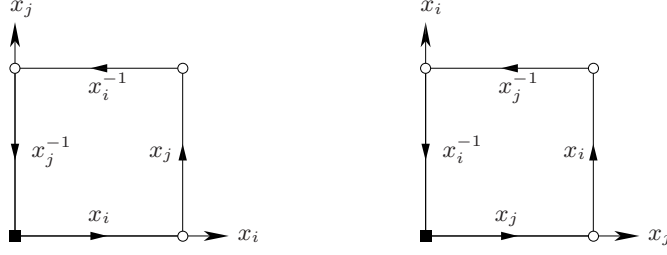
A graph group is given by a finite combinatorial graph Δ ; its vertices x_j are the standard generators of the group $G = G_\Delta$ and its edges $\{x_j, x_h\}$ give rise to the defining relations $x_jx_h = x_hx_j$. Let \mathcal{R} be the set consisting of the commutator $[x_j, x_h] = x_jx_hx_j^{-1}x_h^{-1}$ and its inverse $[x_h, x_j]$ for every edge $\{j, h\}$ of Δ . We contend that $\Psi(\mathcal{R})$ coincides with $\Sigma^1(G)$. The inclusion $\Psi(\mathcal{R}) \subseteq \Sigma^1(G)$ holds by Proposition B2.15; it suffices thus to establish the opposite inclusion. This we do with the help of Theorem B1.17. It shows that if a non-zero character χ represents a point of $\Sigma(G)$ then its living subgraph $\mathcal{L}(\chi)$ is *connected* and every vertex outside $\mathcal{L}(\chi)$ must be *adjacent to a vertex in $\mathcal{L}(\chi)$* .

These conditions are, in essence, requirements (i) and (ii) enunciated in the definition of $\Psi(\mathcal{R})$. To see this, we compare first the living subgraph $\mathcal{L}(\chi)$, defined in section B1.3a, with the graph $\mathcal{G}_{\mathcal{R}}(\chi)$ occurring in definition B2.13. The vertex set of $\mathcal{L}(\chi)$ consists of the generators x_i with non-zero χ -values; its edge set is induced by that of the defining graph Δ , for $\mathcal{L}(\chi)$ is a full subgraph of Δ . The vertex set of the graph $\mathcal{G}_{\mathcal{R}}(\chi)$, on the other hand, is the set of letters

$$\mathcal{Y}_{>0} = \{x_i^{\varepsilon_i} \mid \varepsilon_i \cdot \chi(x_i) > 0 \text{ and } |\varepsilon_i| = 1\}.$$

Two letters y_1, y_2 are connected in $\mathcal{G}_{\mathcal{R}}(\chi)$ if there is a relator $r \in \mathcal{R}$ such that the sequence $f_{(r, \chi)}$ assumes its minimum only once and if the letters involved in the minimum are y_1, y_2 . Now each of the relators in \mathcal{R} contains only two generators and the only relator in which the letters $y_1 = x_i^{\varepsilon_i}$ and $y_2 = x_j^{\varepsilon_j}$ occur are the commutator $[x_i, x_j]$ and its inverse $[x_j, x_i]$. Figure B.9 then allows us to conclude that the graph $\mathcal{G}_{\mathcal{R}}(\chi)$ contains the edge $\{x_i^{\varepsilon_i}, x_j^{\varepsilon_j}\}$ if, and only if the living graph contains the edge $\{x_i, x_j\}$. The two graphs are therefore isomorphic.

Consider now a generator x_j with $\chi(x_j) = 0$. Then x_j is not a vertex of the living graph and so it is, by assumption, an end point of an edge whose other end

Figure B.9: Paths of the relators $[x_i, x_j]$ and $[x_j, x_i]$

point is a vertex x_i , say, of $\mathcal{L}(\chi)$. Since x_j is adjacent to x_i , the commutators $[x_i, x_j]$ and $[x_j, x_i]$ are elements of \mathcal{R} . A glance at Figure B.9 then shows that each of the sequences $f_{[x_i, x_j]}(\chi)$ and $f_{[x_j, x_i]}(\chi)$ assumes its minimum twice and that the letters involved in this minimum are x_i^σ and x_j^ε with σ the sign of $\chi(x_i)$. So the generator $x_j \in \mathcal{X}_0$ satisfies condition (ii) of definition B2.13.

B2.4d Application to knot groups

Let k be an oriented knot in the 3-sphere, represented by an oriented, regular diagram D in the euclidian plane. The crossings of D give rise to arcs $\alpha_1, \dots, \alpha_m$, going from one undercrossing to the next one. The Wirtinger presentation associates to each arc α_j a generators x_j of G and to each crossing a defining relator of the form

$$r_j = x_j \cdot x_{\beta(j)}^{\sigma(j)} \cdot x_{j+1}^{-1} \cdot x_{\beta(j)}^{-\sigma(j)} \text{ for } j \in \{1, \dots, m\}. \quad (\text{B2.23})$$

Here x_{m+1} is to be read as x_1 and $\sigma(j) = \pm 1$. (See [Fox62, pp. 121–122], [Rol76, pp. 56–57] or [BZ03, p. 8 and pp. 32–35] for terminology and proofs.)

The relators imply that G_{ab} is infinite cyclic, and that every character $\chi: G \rightarrow \mathbb{R}$ assume the same value on the generators x_j . The sphere has therefore only two points, represented by χ_+ with $\chi_+(x_j) = 1$ and its negative. Set $\mathcal{R} = \{r_1, \dots, r_m\}$. Each of the graphs $\mathcal{G}_{\mathcal{R}}(\chi_+)$ and $\mathcal{G}_{\mathcal{R}}(\chi_-)$ has m -vertices and m -edges.

The edge set of the first graph is the set of generators and its edges are

$$e_j = \begin{cases} \{x_j, x_{\beta(j)}\} & \text{if } \sigma(j) = 1, \\ \{x_{j+1}, x_{\beta(j)}\} & \text{if } \sigma(j) = -1 \end{cases} \text{ for } j \in \{1, \dots, m\}.$$

The definition of $\mathcal{G}_{\mathcal{R}}(\chi_-)$ is similar. Notice that $[\chi_+] \in \Psi(\mathcal{R})$ precisely if $\mathcal{G}_{\mathcal{R}}(\chi_+)$ is connected, and similarly for $[\chi_-]$.

The graphs $\mathcal{G}_{\mathcal{R}}(\chi_+)$ and $\mathcal{G}_{\mathcal{R}}(\chi_-)$ are easy to work out, but they depend strongly on the chosen diagram. We shall come back to them in a later chapter.

B3 Rank 1 points in Σ^1

Theorem A5.1 in Chapter A describes an obstruction to the finite presentability of a soluble group G . The theorem goes back to Theorem C in [BNS87], but it had a precursor, namely Theorem A in [BS78]. This old result is a *structure theorem* for finitely presented groups admitting a rank 1 character χ and reads like this:

Let G be a finitely presented group and $\chi: G \rightarrow \mathbb{R}$ a rank 1 character. Then G is a HNN-extension with a single stable letter, a finitely generated base group and finitely generated associated subgroups, all three contained in the kernel of χ .

This structure theorem leads to several types of consequences; some of them can not be inferred from Theorem A5.1. Moreover, recent investigations have led to a generalization of the structure theorem; this generalization has resulted in new applications of Σ^1 . These consequences and the generalization call for an adequate treatment of the structure theorem in this monograph on Sigma invariants.

The present section provides such an account. It begins with the statement and proof of the structure theorem, followed by a summary of important consequences of the result. Then three major consequences will be discussed in some detail. The section will close with a description of the generalization alluded to.

B3.1 Structure theorem for indicable groups of type FP_2

Let G be a finitely generated soluble group or, more generally, a *finitely generated group which contains no free subgroup of rank 2*. If G admits a finite presentation and $\chi: G \rightarrow \mathbb{R}$ is a non-zero character, then Theorem A5.1 asserts that $\Sigma^1(G)$ contains at least one of the points $[\chi]$ or $[-\chi]$. In this statement, χ is allowed to be a character of rank greater than 1.

If the rank of χ is 1, the conclusion of Theorem A5.1 is a consequence of the following more general result:

THEOREM B3.1 *Assume G is a finitely generated group that is of type FP_2 over some commutative ring K (with $1 \neq 0$) and $\chi: G \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ is a rank 1 character. Choose an element $t \in G$ with $\chi(t) = 1$ and set $N = \ker \chi$. Then there exist finitely generated subgroups S , T and B of N , with S and T contained in B , so that conjugation by t induces an isomorphism $\mu: S \xrightarrow{\sim} T$ and such that the inclusion of B in G and the assignment $y \mapsto t$ induce an isomorphism*

$$\tau: \langle B, y \mid y \cdot s \cdot y^{-1} = \mu(s) \text{ for all } s \in S \rangle \xrightarrow{\sim} G. \quad (\text{B3.1})$$

Stated in less technical terms, Theorem B3.1 asserts that *every indicable group of type FP_2* is an HNN-extension with finitely generated base group B and finitely generated associated groups; moreover, B can be chosen to be a subgroup of the kernel of a preassigned rank 1 character χ . (Recall that a group is called *indicable* if it projects onto \mathbb{Z} .)

on the Sigma invariants

B3.1a Proof of the structure theorem

In many applications, the group G is finitely presentable, not merely of type FP_2 over some commutative ring. As the proof of Theorem B3.1 simplifies sensibly under this stronger hypothesis, we give first the proof with this stronger hypothesis, and then indicate how the proof can be modified so as to become sound under the weaker hypothesis,

Proof for finitely presented groups. Assume G is a finitely presented group and let $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ be a rank 1 character. Fix $t \in G$ with $\chi(t) = 1$ and choose a finite set of generators b_1, \dots, b_f in $N = \ker \chi$ so that $G = \text{gp}(b_1, \dots, b_f, t)$. Next, let F be the free group on the set $\{x_1, \dots, x_f, y\}$ and ρ the epimorphism of F onto G determined by $x_1 \mapsto a_1, \dots, x_f \mapsto a_f$ and $y \mapsto t$. By a basic result of B. H. Neumann's (see [Neu37, Lemma 8]), the kernel R of the projection $\rho: F \twoheadrightarrow G$ is then the normal closure of a finite set of relators, say r_1, \dots, r_k . Each relator r_j is a product of powers of conjugates $y^\ell x_i y^{-\ell}$; by replacing, if need be, some of these relators by conjugates, we can arrange that only non-negative powers ℓ occur in the new defining relators. Since their number is finite there is a natural number m such that every r_j is contained in the group

$$V = \text{gp}(\{y^\ell x_i y^{-\ell} \mid 0 \leq \ell \leq m+1 \text{ and } 1 \leq i \leq f\}).$$

Define $B \subset G$ to be the image of V under the epimorphism ρ ; so

$$B = \text{gp}(\{t^\ell b_i t^{-\ell} \mid 0 \leq \ell \leq m+1 \text{ and } 1 \leq i \leq f\}),$$

and introduce the groups $S = \text{gp}(\{t^\ell b_i t^{-\ell} \mid 0 \leq \ell \leq m \text{ and } 1 \leq i \leq f\})$ and $T = tSt^{-1}$. Clearly, S and T are subgroups of B .

The inclusion of B in G and the assignment $u \mapsto t$ induce a homomorphism

$$\tau: \tilde{G} = \langle H, u \mid u \cdot s \cdot u^{-1} = \mu(s) \text{ for all } s \in S \rangle \rightarrow G$$

where $\mu(s) = tst^{-1}$. This homomorphism is *surjective*; indeed, the elements b_1, b_2, \dots, b_f, t generate G and every generator b_i lies in B . Next the assignments $x_i \mapsto b_i$, where $1 \leq i \leq f$, and $x \mapsto u$ induce a homomorphism $\nu: F \twoheadrightarrow \tilde{G}$ of the free group F onto the HNN-extension \tilde{G} . The defining relations of \tilde{G} state that

$$u \cdot (t^\ell b_i t^{-\ell}) \cdot u^{-1} = t \cdot (t^\ell b_i t^{-\ell}) \cdot t^{-1} \text{ for } 0 \leq \ell \leq m \text{ and } 1 \leq i \leq f.$$

This implies that $u^\ell b_i u^{-\ell} = t^\ell b_i t^{-\ell}$ in \tilde{G} for all $0 \leq \ell \leq m+1$ and $1 \leq i \leq f$. Consequently, ν maps $r_j = w_j(y^\ell x_i y^{-\ell})$ to $w_j(u^\ell b_i u^{-\ell}) = w_j(t^\ell b_i t^{-\ell}) = 1$, and so ν induces an epimorphism $\nu_*: F/R \twoheadrightarrow \tilde{G}$. The composition

$$\tau \circ \nu_*: F/R \twoheadrightarrow \tilde{G} \twoheadrightarrow G$$

is nothing but the isomorphism induced by ρ and so $\tau: \tilde{G} \rightarrow G$ is an isomorphism.

Notes

Adaptation of the proof for groups of type FP_2 . Assume G is of type FP_2 over the commutative ring K . Then Lemma A5.10 allows one to find a short exact sequence $M \hookrightarrow G_1 \xrightarrow{\pi} G$ of groups in which G_1 is finitely related and $K \otimes_{\mathbb{Z}} M_{\text{ab}} = 0$. Choose a finite set of generators b_1, \dots, b_f, t_1 in G_1 in such a way that $\pi: G_1 \rightarrow G$ maps t_1 onto t and the remaining generators b_i onto elements a_i in the kernel of $\chi: G \rightarrow \mathbb{R}$.

Next define F to be the free group on the set $\{x_1, \dots, x_f, y\}$ and ρ the epimorphism of F onto G_1 determined by $x_1 \mapsto b_1, \dots, x_f \mapsto b_f$ and $y \mapsto t_1$. The kernel R of the projection $\rho: F \rightarrow G_1$ is the normal closure of a finite set of relators, say r_1, \dots, r_k . Each relator r_j is a product of powers of conjugates $y^\ell x_i y^{-\ell}$; by replacing, if need be, some of these relators by conjugates, we can arrange that only non-negative powers ℓ occur in the new defining relators. Since their number is finite there is a natural number m such that every r_j is contained in the group

$$V = \text{gp}(\{y^\ell x_i y^{-\ell} \mid 0 \leq \ell \leq m+1 \text{ and } 1 \leq i \leq f\}).$$

Define $B \subset G$ to be the image of V under the composition $\pi \circ \rho$; so

$$B = \text{gp}(\{t^\ell a_i t^{-\ell} \mid 0 \leq \ell \leq m+1 \text{ and } 1 \leq i \leq f\}),$$

and introduce the groups $S = \text{gp}(\{t^\ell a_i t^{-\ell} \mid 0 \leq \ell \leq m \text{ and } 1 \leq i \leq f\})$ and $T = tSt^{-1}$. Clearly, S and T are subgroups of B .

The inclusion of B in G and the assignment $u \mapsto t$ induce a homomorphism

$$\tau: \tilde{G} = \langle B, u \mid u \cdot s \cdot u^{-1} = \mu(s) \text{ for all } s \in S \rangle \rightarrow G;$$

as before, $\mu(s) = tst^{-1}$. This homomorphism is *surjective*. Next the assignments $x_i \mapsto a_i$, where $1 \leq i \leq f$, and $x \mapsto u$ induce a homomorphism $\nu: F \rightarrow \tilde{G}$ of the free group F onto the HNN-extension \tilde{G} . It then follows, exactly as in the previous proof, that ν induces an epimorphism $\nu_*: G_1 = F/R \rightarrow \tilde{G}$. The composition

$$\tau \circ \nu_*: G_1 \rightarrow \tilde{G} \rightarrow G$$

is nothing but π and so its kernel is M . This kernel M is mapped by ν_* onto the kernel of τ . As $K \otimes_{\mathbb{Z}} M_{\text{ab}} = 0$, it follows that $K \otimes_{\mathbb{Z}} (\ker \tau)_{\text{ab}} = 0$. On the other hand, τ maps the base group B identically onto $B \subset G$ and thus its kernel is a free group ([KS71, Corollary 1]; cf. [Coh89, p. 212, Corollary 2]). But if so, $\ker \tau$ must be trivial, whence $\tau: \tilde{G} \rightarrow G$ is an isomorphism.

NOTE B3.2 a) Theorem B3.1 and its proof go back to [BS78, Thm. A]; the given, more detailed, version of the proof is taken from the review [Str84].

b) In Theorem B3.1 one considers a finitely presented group G that maps onto an infinite cyclic group with kernel N and deduces that G can be written as an HNN-extension with finitely generated base group and finitely generated associated subgroups contained in N . In [BM07], G. Baumslag and C. Miller III generalize this result to a theorem involving HNN-extensions with several stable letters; their theorem reads as follows ([BM07, Thm. 1]): *Let G be a finitely presented group which projects onto a free group of rank n ; let N denote the kernel of the projection. Then G is an HNN extension with n stable letters, of a finitely generated group B and finitely generated associated subgroups, all contained in N .*

B3.1b Reflections on the consequences of the structure theorem

In the remainder of Section B3, four consequences of Theorem B3.1 will be discussed. The first one is straightforward: given an indicable group G of type FP_2 one infers from the structure theorem that G splits as an HNN-extension over a finitely generated subgroup, and uses this information to deduce further properties of G . Whether this approach leads to useful insights depends, of course, on additional properties of the group G . Some situations where this approach has led to satisfactory results will be described in section B3.1c.

The next two applications make use of a *dichotomy in the class of HNN-extensions*; there are *ascending* HNN-extensions, here the base group coincides with one of the associated subgroups, and *non-degenerate* HNN-extensions where the base group is distinct from both associated subgroups. This dichotomy can be used as follows.

As before, let G be an indicable group of type FP_2 (over some commutative non-zero ring) and let χ be a fixed rank 1 homomorphism. In one application, one assumes that G cannot be written as a non-degenerate HNN-extension³ or, less restrictive, as a non-degenerate HNN-extension with finitely generated base group and associated subgroups, all contained in the kernel N of χ . The structure theorem then implies that G is an ascending HNN-extension with finitely generated base group contained in N ; or, alternatively, that $\Sigma^1(G)$ contains (at least) one of the points $[\chi]$ or $[-\chi]$. This conclusion is an analogue of the conclusion of Theorem A5.1 applied to the character χ .

In the other application, one assumes that G is not an ascending HNN-extension with finitely generated base group contained in the kernel of χ or, in other words, that $[\chi]$ and $[-\chi]$ lie both outside of $\Sigma^1(G)$ and one deduces that G is a non-degenerate HNN-extension with finitely generated base group and associated subgroups, all three contained in the kernel of χ .

There is a forth application. It is a contraposition of the structure theorem. This time, G is a finitely generated group and $\chi: G \rightarrow \mathbb{R}$ is a cleverly chosen rank 1 character. One assumes that G cannot be written as an HNN-extension with finitely generated base group and finitely generated associated subgroups, all three contained in the kernel of χ . The structure theorem then implies that G is *infinitely related* and that it is not of type FP_2 , no matter how the commutative ring is chosen.

REMARKS B3.3 a) If the sphere $S(G)$ has positive dimension, there are infinitely many rank 1 points and hence infinitely many essentially different choices for χ . In some of the applications, one uses this infinite set of choices, in others they are of little significance. In the first application, one is interested in a splitting of G as an HNN-extension with finitely generated subgroups B , S and T , and the choice of χ will typically play no rôle in the subsequent analysis. If the aim is to show that $\Sigma^1(G)$ contains at least one of the points $[\chi]$ or $[-\chi]$, one often varies

³This assumption holds, for instance, if G does not contain a non-abelian free subgroup.

$[\chi]$ so as to find many points in $\Sigma^1(G)$. In the other two applications, a single well chosen character will lead to the conclusion that G admits a non-degenerate HNN-decomposition and hence contains a non-abelian free subgroup, respectively that G is infinitely related.

b) As stated before, the forth application is a contraposition of the structure theorem. It has come to light recently that one can do better: by going through the proof of the structure theorem one can derive additional properties of the group G (see [BCGS12, Thm. 6.1]). In section B3.5 I shall briefly explain the additional insights afforded by the new approach.

B3.1c Direct applications of the structure theorem

Recall the succinct version of Theorem B3.1: *every indicable group of type FP_2 is an HNN-extension with finitely generated base group B and finitely generated associated groups; moreover, B can be chosen to be a subgroup of the kernel of a preassigned rank 1 character χ .*

In the literature, one can find several instances where this structure theorem is applied directly. Often, however, it is difficult to describe the use of the theorem without recalling a fair number of details from the context of the application. In what follows, I sketch two applications that need little preparations.

Poincaré duality groups of dimension 2. The first and earliest application can be found in the paper [EM80] by B. Eckmann and H. Müller. The aim of this paper is to show that *an indicable Poincaré group G of dimension 2 is a surface group*. A first problem on the path towards this goal is the fact that the hypothesis on the homology and cohomology of G implies only that the group is of type FP_2 over \mathbb{Z} (by [BE74], cf. [Bro75] or [Str76]), while a surface group has a one-relator presentation. The authors proceed as follows.

Assume G is an indicable Poincaré duality group of dimension 2. Then G is of type FP_2 over \mathbb{Z} and so it is an HNN-extension with finitely generated base group and associated subgroups (by Theorem B3.1). As the base group B has infinite index in G , it is of cohomological dimension 1 (by [Str77]) and hence free by Stallings's Theorem. If the rank of B is greater than 1 the splitting can be changed so as to become a splitting of G , as a free product with amalgamation or as an HNN-extension, over a subgroup of smaller rank (by the *decomposition theorems for group pairs* established in [Mül81]). One is thus reduced to the case where the amalgamated subgroup A or the the associated subgroup S is infinite cyclic. Then the group pairs $(G_1; A)$ and $(G_2; A)$ in the case of an generalized free product, or the pair $(B; \{S, T\})$ in the case of an HNN-extension, are PD^2 -pairs (see [BE78]). By [EM80, Thm. 2] these PD^2 -pairs are geometric. It then follows easily that $G = G_1 \star_A G_2$ or $G = \langle B, t \mid tSt^{-1} = T \rangle$ is a surface group.

Structure of finitely presented LERF groups. A group G is said to be *LERF* (short for *locally extended residually finite*) or *subgroup separable* if every *finitely generated* subgroup H of G is an intersection of finite index subgroups of G . Examples of

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such groups are free groups ([Hal49, Thm. 5.1]), polycyclic groups (see [Rob96, 5.4.16]) and fundamental groups of closed surfaces ([Sco78, Thm. 3.3]).

A LERF group is, of course, residually finite, but the converse does not hold. This fact is made clear by the main result of [BN74], asserting that *a subgroup H that is an intersection of subgroups of finite index cannot be conjugated to a proper subgroup of itself*. (This implies, in particular, that the Baumslag-Solitar $BS(1, m)$ with $m > 1$ are not LERF.)

In [But08], J. O. Button establishes the following property of finitely presented LERF groups:

PROPOSITION B3.4 ([BUT08, THEOREM 7.3]) *A finitely presented LERF group is large⁴ or all the kernels of its rank 1 characters are finitely generated.*

Proof. As the conclusion is clearly true if G_{ab} is finite, we assume henceforth that G is indicable and consider a rank 1 character $\chi: G \rightarrow \mathbb{R}$. The structure theorem then shows that G is an HNN-extension of the form

$$\langle B, y \mid ysy^{-1} = \mu(s) \text{ for all } s \in S \rangle$$

with finitely generated base group B contained in $N = \ker \chi$ and finitely generated associated subgroups $S, T = \mu(S)$. Since G is LERF this HNN-extension is neither properly ascending nor properly descending (by [BN74, Theorem]), and so only two possibilities remain: either $S = T = B$, or $S \subsetneq B$ and $T \subsetneq B$. In the first case, B is normal in G , hence coincides with $N = \ker \chi$ and so N is finitely generated.

In the second case, we choose $b_0 \in B \setminus S$ and then use the hypothesis that G be LERF to find a finite index subgroup H such that $H \supset S$ and $b_0 \notin H$. Define $L = \bigcap_{g \in G} gHg^{-1}$ to be the core of H in G . Then L is a normal subgroup of G with finite index; moreover, $S \cdot L \subseteq H$ and so $S \cdot L \subsetneq B \cdot L$. It follows that the canonical projection $\pi: G \twoheadrightarrow \bar{G} = G/L$ maps S onto a proper subgroup \bar{S} of $\bar{B} = \pi(B)$. The associated subgroup T maps onto a subgroup \bar{T} of \bar{B} ; as it is conjugated to S , its image \bar{T} is conjugated to \bar{S} and so distinct from \bar{B} , for $\text{card}(\bar{S}) < \text{card}(\bar{B}) < \infty$. All taken together, this shows that $\bar{G} = G/L$ is a non-degenerate HNN-extension

$$\langle \bar{B}, \bar{y} \mid \bar{y}\bar{S}\bar{y}^{-1} = \bar{T} \rangle$$

with finite base group. Such an HNN-extension is large (this follows, e. g., from [Ser77, II, §2, Prop. 11] or from [SW79, Lemma 7.4], and the fact that \bar{G} contains infinitely generated subgroups), whence G is large. \square

B3.2 Applications to finitely presented soluble groups

The next consequence of Theorem B3.1 is the *HNN-criterion for finitely presented groups G containing no free subgroups of rank 2*. The criterion reads as follows:

⁴Following Gromov in [Gro82, p. 82, Thm. (B)], a group will be called *large* if it has a subgroup of finite index that maps onto a free subgroup of rank 2.

PROPOSITION B3.5 (HNN-CRITERION) *Assume G is a group of type FP_2 which does not contain a non-abelian free subgroup and whose abelianization has positive rank. Let $\chi: G \rightarrow \mathbb{R}$ be a rank 1 character. Then G is an ascending HNN-extension with finitely generated base group $B \subseteq \ker \chi$; put differently, $\Sigma^1(G)$ contains (at least) one of the points $[\chi]$ or $[-\chi]$.*

Proof. The hypotheses of Proposition B3.5 include the assumptions of Theorem B3.1 and so the conclusion of the latter result shows that G is an HNN-extension with finitely generated subgroup B contained in $N = \ker \chi$ and finitely generated associated subgroups S and T . This HNN-extension cannot be non-degenerate for otherwise N would contain non-abelian free subgroups by Lemma B3.6 below, in contradiction to the hypothesis of Proposition B3.5. So at least one of the equalities $S = B$ or $T = B$ holds, i. e., the HNN-ascension is descending or ascending. By Proposition A3.4 this finding implies that at least one of the points $[-\chi]$ or $[\chi]$ lies in $\Sigma^1(G)$. \square

LEMMA B3.6 *Assume G is an HNN-extension with stable letter t and associated subgroups S, T both distinct from the base group B . Then G contains a non-abelian free subgroup.*

Proof. Since $S \subsetneq H$ and $T \subsetneq H$, there exist elements $a \in H \setminus S$ and $b \in H \setminus T$. Then $x = at^{-1}$ and $y = bt$ generate a subgroup of G ; it is free on $\{x, y\}$ by Britton's Lemma (see, e. g., [LS01, p. 181]). \square

B3.2a Remark on Σ^1 and non-degenerate HNN-extensions

If one applies the proof of Theorem B3.1 to a group given by an explicit finite presentation, one obtains finite generating systems for B , S and T . One may then be able to find out by direct inspection whether $S = B$ or $T = B$. In the first case, $[-\chi] \in \Sigma^1(G)$, in the second case, one has $[\chi] \in \Sigma^1(G)$.

Suppose now that $S \neq B$ or $T \neq B$. Then N must be infinitely generated, whence Corollary A4.3 shows that at least one point in the antipodal pair $\{[\chi], -\chi\}$ lies outside of $\Sigma^1(G)$. One can do better:

PROPOSITION B3.7 *Let G be a finitely generated group that can be written as an HNN-extension as given in equation (B3.1), the map $\mu: S \xrightarrow{\sim} T$ being the restriction of the inner automorphism $g \mapsto tgt^{-1}$ induced by $t = \tau(y) \in G$.*

Let $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ denote the character that sends t to 1 and maps the base group B to $\{0\}$. If B is finitely generated, the following implication holds:

$$T \neq B \implies [\chi] \notin \Sigma^1(G) \tag{B3.2}$$

Proof. Let $\eta: \mathcal{Z} \rightarrow B$ be a finite generating system of the base group B and set $\mathcal{X} = \mathcal{Z} \cup \{t\}$. Then \mathcal{X} generates G and so the Cayley graph $\Gamma = \Gamma(G, \mathcal{X})$

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is connected. Consider now a path p with origin 1 that runs in the subgraph $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$. This path can be written as $p = (1, w)$ where w is a word of the form

$$t^{e_0} w_1 t^{e_1} w_2 t^{e_2} \dots t^{e_{k-1}} w_k t^{e_k}$$

with each subword w_j a word in \mathcal{Z}^\pm , each exponent e_j in $\{1, -1\}$, and each partial sum $e_0 + \dots + e_j$ non-negative. If w contains a subword of the form $tw_j t^{-1}$ with $b = \eta_*(w_j) \in S$, we replace the subword by a word w'_j in \mathcal{Z}^\pm that represents $\mu(h)$. The resulting path w' will then again run in the subgraph Γ_χ . If w contains a subword of the form $t^{-1}w_j t$ with $\eta_*(w_j) \in T$, one proceeds similarly. There exists therefore a word w' with $\eta_*(w') = \eta_*(w)$ such that $p' = (1, w')$ runs in Γ_χ and that the associated sequence

$$(t^{e_0}, b_1, t^{e_1}, b_2, t^{e_2}, \dots, t^{e_{\ell-1}}, b_\ell, t^{e_\ell}) \quad \text{with} \quad b_j = \eta_*(w_j) \quad (\text{B3.3})$$

is *reduced*, i.e., it contains neither a subsequence (t, b_j, t^{-1}) with $b_j \in S$ nor a subsequence (t^{-1}, b_j, t) with $b_j \in T$.

We are now ready to show that Γ_χ is not connected, whence $[\chi] \notin \Sigma^1(G)$ by Theorem A2.3. By assumption, there exists $b_0 \in B \setminus T$. The product $g_0 = t^{-1}b_0t$ is a vertex of Γ_χ ; if it could be connected to 1 inside Γ_χ , the above reasoning would provide us with a reduced sequence of the form (B3.3) with product equal to g_0 . As this reduced sequence would correspond to a path in Γ_χ , one would have $\ell > 0$ and $e_0 \geq 0$ and so the sequence

$$(t^{-1}, b_0^{-1}, t^{1+e_0}, b_1, t^{e_1}, \dots, t^{e_{\ell-1}}, b_\ell, t^{e_\ell})$$

would be reduced. As the product of this sequence would be the unit element, it would contradict Britton's lemma (see, e.g., [LS01, p. 181]). \square

NOTE B3.8 Proposition B3.7 goes back to [BNS87, Prop. 4.4]. It will be generalized by part (ii) of Proposition C2.13; the generalization will reveal that the hypothesis that B be finitely generated is redundant.

B3.2b Application to finitely presented nilpotent-by-cyclic groups

We continue with applications of Proposition B3.5 to some selected classes of finitely related soluble groups. We start out with nilpotent-by-cyclic groups.

The class of *finitely generated* nilpotent-by-cyclic groups contains the centre-by-metabelian 3-generator subgroup

$$G = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \quad (\text{B3.4})$$

of $\text{GL}(3, \mathbb{Q}(X))$ with X an indeterminate, and the class contains all homomorphic images of G . It follows that the groups in the class are in general neither residually finite, nor do they satisfy max-n, the maximal condition on normal subgroups. These facts are already pointed out in [Hal54].

The situation is entirely different for *finitely related* nilpotent-by-cyclic groups:

COROLLARY B3.9 *Assume G is a finitely generated extension of a nilpotent group N by a cyclic group G/N . Then G is finitely related if, and only if, it is polycyclic or else an ascending HNN-extension whose base group is finitely generated and nilpotent.*

Proof. Assume G is finitely related and let $N \triangleleft G$ be a nilpotent normal subgroup with cyclic quotient G/N . If G/N is finite, then G is polycyclic. Otherwise, G is by the HNN-criterion an ascending HNN-extension $\langle B, t \mid t^{-1}Bt \subseteq B \rangle$ whose base group B is finitely generated and nilpotent. The converse claim is clearly true. \square

REMARKS B3.10 a) Corollary B3.9 is result 11.4.5 in [LR04] and goes back to Theorem C in [BS78]. It shows, first of all, that the class of finitely presented nilpotent by infinite cyclic groups admits of an explicit description, in contrast to the class of finitely generated nilpotent by infinite cyclic groups.

Moreover, the corollary implies that a finitely presented nilpotent-by-infinite cyclic group satisfies max-n ([BB76, Cor. 5], cf. [LR04, 11.4.4 and 11.2.6]), is residually finite ([LR04, 11.4.4 and 11.2.4]), cf. [BB76, Cor. 7]), that the torsion subgroup of its nilpotent normal subgroup N is finite and the group has finite Hirsch length and that it is coherent ⁵ ([BS79, Thm. B] or [Gro78b], cf. [LR04, 11.4.4]).

b) Suppose G is a finitely related nilpotent by infinite cyclic group. Then Corollary B3.9 describes the structure of G and it implies that G satisfies a number of useful properties, but the corollary does not provide a classification of the groups in question. One reason is this: given a finitely generated nilpotent group, one would like to know all ascending HNN-extensions having B as base group. This presupposes, in particular, that one can describe the injective endomorphisms of B . Some information on this problem can be found on page 261 of [LR04].

EXAMPLE B3.11 Let p and q be positive and relatively prime integers and let $G_{p,q}$ denote the semi-direct product $\mathbb{Z}[1/(p \cdot q)] \rtimes \text{gp}(t)$ with t is an element of infinite order that acts on $A = \mathbb{Z}[1/(m \cdot n)]$ by multiplication by m/n . (See Example A3.6 for more details.) A finitely generated subgroup of A is cyclic and so G is an ascending HNN-extension with finitely generated base group if, and only if, $m = 1$ or $n = 1$. Corollary B3.9 thus implies that G is finitely related precisely if it is one of the Baumslag-Solitar groups $\text{BS}(m, 1)$ or $\text{BS}(1, n)$.

NOTE B3.12 At the beginning of the 1970s both R. Bieri and J. Conway stumbled over the question whether the group $G_{2,3}$ is infinitely related. G. Higman answered John's question in the affirmative, as did G. Baumslag in 1973 when asked by Robert (cf. [BS76, (1.11)]). The determination of the finitely related specimens among the groups $G_{p,q}$ predates [BS78] and is contained in [BS76] (use Lemma C on p.57). The fact that $G_{p,q}$ is finitely related if, and only if, it is of type

⁵A group is called coherent if each of its finitely generated subgroups is finitely related.

FP_2 follows from [BS78] and is used in the classification of the soluble groups of cohomological dimension 2 published by D. Gildenhuys in [Gil79].

EXAMPLE B3.13 Let G be a *finitely generated, abelian by infinite cyclic group*. Then G is, of course, metabelian and a semi-direct product $A \rtimes C$ of an abelian group A by an infinite cyclic group $C = \text{gp}(p)$. The wreath products $(\mathbb{Z}/k\mathbb{Z}) \wr C$ and $\mathbb{Z} \wr C$ make it clear that the torsion subgroup $\text{tor}(A)$ of A can be infinite and that $\bar{A} = A/\text{tor}(A)$ can have infinite torsion-free rank. Conjugation by the elements of C turns A into a $\mathbb{Z}C$ -module and as such A is a finitely presentable $\mathbb{Z}C$ -module; in other words, A has a presentation of the form

$$\mathbb{Z}C^m \xrightarrow{\partial} \mathbb{Z}C^n \xrightarrow{\pi} A \rightarrow 0$$

where ∂ is given by multiplication on the right by an $m \times n$ matrix with entries in $\mathbb{Z}C$. (Special presentations of this kind have been studied in classical *Knot Theory*; see, e. g., [Cro63] and [Tro74]).

Suppose now that the group $G = A \rtimes C$ is *finitely related*. Then the torsion subgroup of A is finite and $\bar{A} = A/\text{tor}(A)$ is a torsion-free group of finite torsion-free rank d , say. Moreover, for a suitable generator $t \in C$, the $\mathbb{Z}C$ -module \bar{A} contains a free-abelian subgroup \bar{B} of rank d with $t.\bar{B} \subseteq \bar{B}$ and $\bar{A} = \bigcup_{j < 0} t^j.\bar{B}$.

It follows that each torsion-free, finitely presented abelian by infinite cyclic group is given by an integer square matrix M with non-zero determinant.

NOTE B3.14 The fact that a finitely presented abelian by infinite cyclic group G can be described by an integer square matrix M is the starting point of several papers dealing with the large-scale geometry of finitely presented abelian-by-cyclic groups that are *not* polycyclic (see, e. g., [FM98], [FM99], [FM00]). Similarly, the characterization of finitely presented nilpotent-by-cyclic groups, afforded by Corollary B3.9, is used in [Ahl05].

B3.2c Application to coherent soluble groups

A group is termed *coherent* if every finitely generated subgroup is finitely presented. Obvious examples of coherent groups are free groups, locally finite groups and polycyclic-by-finite groups. On the other hand, finitely presented metabelian groups are typically not coherent. This statement can be illustrated by the following simple example.

EXAMPLE B3.15 Let Q a free-abelian group of rank 2, generated by s, t , and let p and q be relatively prime positive integers. Set $A = \mathbb{Z}[1/(p \cdot q)]$ and turn A into a $\mathbb{Z}Q$ -module by declaring that s act by multiplication by p and t by multiplication by q . Finally, put $G = G_{p,q} = A \rtimes Q$. Then G is a finitely related, metabelian group with presentation $\langle a, s, t \mid sas^{-1} = a^p, tat^{-1} = a^q \text{ and } st = ts \rangle$.

Notes

Every couple $(m, n) \in \mathbb{Z}^2$ gives then rise to a 2-generator subgroup

$$H_{m,n} = \text{gp}(a = (1_A, 1_Q), u_{m,n} = s^m t^n). \quad (\text{B3.5})$$

Here 1_A denotes the unit element of the ring $A = \mathbb{Z}[1/(p \cdot q)]$ and 1_Q the neutral element of the group Q . The subgroup $H_{m,n}$ is finitely presented whenever $m \cdot n \geq 0$, but it is infinitely related if $m \cdot n$ is negative (see Example B3.11).

There exists a very satisfying characterization of finitely generated coherent soluble groups. It is a consequence of Proposition B3.5 and some powerful results about the structure of finitely generated soluble groups.

THEOREM B3.16 ([GRO78C], [BS79]) *A finitely presented soluble group is coherent if, and only, if it is polycyclic or an ascending HNN-extension with polycyclic base group.*

Sketch of proof. By [Hal54], polycyclic groups are finitely related. As subgroups of polycyclic groups are again so, polycyclic groups are coherent. Assume now G is an ascending HNN-extension $\langle B, t \mid t^{-1} B t \subseteq B \rangle$ with polycyclic base group B and stable letter t . The normal closure N of B in G coincides then with the ascending union $\bigcup \{t^j B t^{-j} \mid j \in \mathbb{N}\}$

Let $G_1 \subseteq G$ be a finitely generated subgroup. Two cases arise: if $G_1 \subseteq N$ then G_1 is contained in a conjugate of B , hence polycyclic and therefore finitely related. Otherwise, set $N_1 = G_1 \cap N$. Then $G_1/N_1 \xrightarrow{\sim} (G_1 \cdot)N/N \subseteq G/N$ is infinite cyclic, say generated by $t_1 = x \cdot t^k$ with $x \in N$. As N is an ascending union of copies of B there exists $\ell \in \mathbb{N}$ with $x \in t^\ell H t^{-\ell}$. Since G is also an ascending HNN-extension with base group $t^\ell H t^{-\ell}$; there is no harm in assuming that $x \in B$. But if so, we may as well set $x = 1$. Set $B_1 = B \cap H$ and $N_1 = N \cap H$. Then B_1 is polycyclic and

$$t_1^{-1} B_1 t_1 = t^{-k} (B \cap G_1) t^k = t^{-k} B t^k \cap t^{-k} G_1 t^k \subseteq B \cap G_1 = B_1.$$

Moreover,

$$N_1 = \bigcup_{j \in \mathbb{N}} (t^k)^j B (t^k)^{-j} \cap G_1 = \bigcup_{j \in \mathbb{N}} (t_1^j B t_1^{-j} \cap t_1^j G_1 t_1^{-j}) = \bigcup_{j \in \mathbb{N}} t_1^j B_1 t_1^{-j}.$$

These calculations show that G_1 is an ascending HNN-extension with polycyclic base group B_1 and hence finitely related.

So far we know that polycyclic groups and ascending HNN-extensions with polycyclic base groups are coherent. The proof of the converse statement is more involved. One has to show, in particular, that a finitely generated coherent soluble group is nilpotent-by-abelian-by-finite (as is every polycyclic group). For details, see pages 236–238 in [BS79] or pages 259–261 in [LR04]. \square

REMARK B3.17 The statement of Theorem B3.16 remains valid if the requirement that G be coherent is replaced by the condition that every finitely generated subgroup of G be of type FP_2 ; see [BS79, Proposition].

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B3.2d Application to finitely presented centre-by-metabelian groups

The group described in equation (B3.4) is a centre-by-metabelian group whose centre is free abelian of rank \aleph_0 . This example permits one to see that the centre of a *finitely generated* centre-by-metabelian group can be any non-trivial countable abelian group. The situation is, however, entirely different for *finitely presented* centre-by-metabelian groups. Indeed, the following important result, published in 1978 by J. R. J. Groves (see [Gro78a, Thm. 1]), holds:

THEOREM B3.18 *A finitely presented centre-by-metabelian group is abelian-by-polycyclic. In particular, it has the maximal condition on normal subgroups and is residually finite.*

Sketch of proof. Let G be a finitely presented centre-by-metabelian group. Then the derived group G' is nilpotent of class at most 2; let $Z = \zeta(G')$ be the centre of G' and set $M = G'/Z$. Suppose we have established that M is a finitely generated (abelian) group. Then G is an extension of an abelian normal subgroup Z by the polycyclic quotient group G/Z ; it satisfies therefore the maximal condition on normal subgroups (by [Hal54, Thm.3], cf. [LR04, 4.2.2]) and is residually finite (by [Jat74, Thm.3] or [Ros76], cf. [LR04, 7.2.1]).

Now to the proof that $M = G'/Z$ is finitely generated. The group A has the structure of a $\mathbb{Z}G_{\text{ab}}$ -module; as such it is finitely generated (for G_{ab} is finitely related). To pin down its structure, one has to exploit the hypothesis that G be finitely related. The idea is to translate the conditions obtained by applying the conclusion of Proposition B3.5 to all rank 1 characters $\chi: G \rightarrow \mathbb{R}$ into a statement about the $\mathbb{Z}G_{\text{ab}}$ -module M . In the account given in section 5.4 of [Str84], the translated statement is expressed in terms of the invariant $\Sigma_M(G_{\text{ab}}) = \Sigma^0(G_{\text{ab}}; M)$. As this invariant will only be discussed in Section D1, we postpone the proof that M is finitely generated *as an abelian group* to this section. \square

NOTE B3.19 Historically speaking, Theorem B3.18 is the first result where a feature of Proposition B3.5 is exploited that is not spelled out explicitly in its statement: if the abelianisation of a finitely presented soluble group G is greater than 1, there are infinitely many pairs $\{\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}, -\chi\}$ of rank 1 characters to which Proposition B3.5 applies.

B3.3 Finding non-abelian free subgroups in fp groups

We come now to a third consequence of Theorem B3.1. Recall that this theorem asserts, loosely speaking, that *every indicable group of type FP_2* is an HNN-extension with finitely generated base group B and finitely generated associated groups. In section B3.2, the HNN-extension has been forced to be ascending by imposing the condition that G contain no non-abelian free subgroups. In this section, we require the HNN-extension to be non-degenerate and conclude that G must contain non-abelian free subgroups.

Notes

Here is the consequence of Theorem B3.1 that will be the basis of our discussion.

PROPOSITION B3.20 *Suppose G is a finitely presented group which admits a rank 1 character $\chi: G \rightarrow \mathbb{R}$ so that G is not an ascending HNN-extension with finitely generated base group B contained in $\ker \chi$. Then G contains non-abelian free subgroups.*

We shall give several applications of the preceding proposition. Each of them presupposes that one knows a rank 1 character $\chi: G \rightarrow \mathbb{R}$ such that G is not an ascending HNN-extension with finitely generated base group contained in $\ker \chi$ or, in terms of the invariant Σ^1 , such that

$$\{[\chi], [-\chi]\} \subseteq \Sigma^1(G)^c. \quad (\text{B3.6})$$

We shall find characters satisfying condition (B3.6) with the help of the following simple, but surprisingly widely applicable

LEMMA B3.21 *Let G be a finitely generated group and $\chi: G \rightarrow \mathbb{R}$ a rank 1 character with kernel N . Assume G contains a normal subgroup $M \subset N$ such that N/M is an infinite, locally finite group. Then $\{[\chi], [-\chi]\} \subseteq \Sigma^1(G)^c$.*

Proof. Let $t \in G_\chi \setminus N$ be an element that generates a complement of N in G and consider a finitely generated subgroup $B \subseteq N$ with $B \subseteq tBt^{-1}$. The image \bar{B} of B under the canonical projection $\pi: G \twoheadrightarrow G/M$ is then finite and hence $\bar{B} \subseteq \pi(t)\bar{B}\pi(t)^{-1}$ implies that $\bar{B} = \pi(t)\bar{B}\pi(t)^{-1}$. It follows that $\bigcup \{t^\ell Bt^{-\ell} \mid \ell \in \mathbb{N}\}$ maps onto \bar{B} . But, by hypothesis, $\bar{B} \neq \bar{N}$ and so $\bigcup \{t^\ell Bt^{-\ell} \mid \ell \in \mathbb{N}\} < N$.

The preceding argument shows that G is not an ascending HNN-extension with finitely generated base group contained in N and stable letter t . Similarly, one sees that G is not an ascending HNN-extension with finitely generated base group contained in N and stable letter t^{-1} . The stated conclusion now follows from Proposition A3.4. \square

B3.3a Application to groups with non-negative deficiency

The preceding lemma and Proposition B3.20 allow one to show that a finitely presented group G with $m \geq 2$ generators and $n \leq m - 2$ relators contains a non-abelian free subgroup. If $n = 1$, i. e., if G is a one-relator group, this fact is an obvious consequence of the *Freiheitssatz* (see, e. g., [LS01, p. 198]). The lemma yields actually a further result: it implies that the invariant $\Sigma^1(G)$ is empty whenever G is a finitely presented group with deficiency greater than 1. In what follows, we first establish this second result and deduce then the first claim from Proposition B3.20.

We begin by recalling the notion of deficiency of a finite presentation.

DEFINITION B3.22 Let $\mathcal{P} = \langle \mathcal{X} \mid \mathcal{R} \rangle$ be a finite presentation of a finitely presentable group G . The difference $\text{card}(\mathcal{X}) - \text{card}(\mathcal{R})$ is called the *deficiency* of \mathcal{P} ; it will be noted $\text{def } \mathcal{P}$. The supremum

$$\text{def } G = \sup\{\text{def } \mathcal{P} \mid \mathcal{P} \text{ is a finite presentation of } G\} \quad (\text{B3.7})$$

is called the *deficiency* of G and denoted by $\text{def } G$.

The supremum in equation B3.7 is always a maximum. Indeed, a finite presentation \mathcal{P} of the group G gives rise to a presentation of the abelianized group G_{ab} ; therefore G_{ab} is a quotient of a free abelian group of rank $\text{card}(\mathcal{X})$ modulo a subgroup generated by $\text{card}(\mathcal{R})$ elements. The torsion-free rank $r_0(G_{\text{ab}}) = \dim_{\mathbb{Q}}(G_{\text{ab}} \otimes \mathbb{Q})$ of G_{ab} is thus an upper bound of $\text{def } \mathcal{P}$.

The deficiency $\text{def } G$ is bounded by $r_0(G_{\text{ab}})$, but this bound need not be optimal. An upper bound that is often smaller is

$$\text{def } G \leq r_0(G_{\text{ab}}) - d(\text{H}_2(G, \mathbb{Z})). \quad (\text{B3.8})$$

In this formula $d(A)$ denotes the minimal number of generators of the group A . (See, e. g., [Rob96, p. 419] for a proof of inequality (B3.8).)

We come now to the announced consequence of Lemma B3.21, actually to a generalization. It asserts, roughly speaking, that the invariant Σ^1 of the metabelian top of a finitely related group with few relators is empty. The precise version of the generalization is spelled out by the next proposition.

PROPOSITION B3.23 *Let G be a finitely related group that admits a presentation with $m \geq 2$ generators and n relators. Fix a prime number p .*

- (i) *If $n \leq m - 2$, or*
- (ii) *if $n = m - 1$ and one relator is a proper power, say $r = w^k$, and p divides k , or*
- (iii) *if $n = m$ and two relators are proper powers, say $r_1 = w_1^{k_1}$ and $r_2 = w_2^{k_2}$, and if p divides both k_1 and k_2 ,*

then $\Sigma^1(Q)$ is empty for every quotient group $Q = G/M$ with $M \subseteq G'' \cdot (G')^p$.

Moreover, if $S(G)$ is non-empty G maps onto the wreath product $(\mathbb{Z}/p\mathbb{Z}) \wr C_\infty$ of a cyclic group of order p by an infinite cyclic group.

Proof. The proof divides into three parts. We first verify that $\Sigma^1(G)$ contains no point of rank 1 and then deduce that the same is true for $\Sigma^1(Q)$. In the final part we invoke the openness of $\Sigma^1(Q) \subseteq S(Q)$ and deduce that $\Sigma^1(Q)$ is empty.

Let $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ be a rank 1 character of G , set $N = \ker \chi$ and pick $t \in G$ with $\chi(t) = 1$. Given a prime number p , consider the elementary abelian p -group $\bar{N} = N/(N' \cdot N^p)$, viewed as a left $\mathbb{F}_p(\text{gp}(t))$ -module via conjugation. Next, let

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$\eta: F \twoheadrightarrow G$ be the epimorphism involved in the presentation with m generators and n relators. Set $R = \ker \eta$ and $U = \eta^{-1}(N)$. Then η gives rise to an extension

$$\bar{R} = (R \cdot U' \cdot U^p)/(U' \cdot U^p) \twoheadrightarrow \bar{U} = U/(U' \cdot U^p) \xrightarrow{\eta^*} \bar{N}$$

of $\mathbb{F}_p(\text{gp}(t))$ -modules. Its middle term \bar{U} is a free $\mathbb{F}_p(\text{gp}(t))$ -module of rank $m - 1$.

Indeed, the matrix group $\text{GL}(m, \mathbb{Z})$ is generated by elementary matrices and each automorphism of $F_{\text{ab}} \xrightarrow{\sim} \mathbb{Z}^m$ that is induced by an elementary matrix lifts to an automorphism of F (cf. the proof of Theorem 3.5 in [MKS04, Chapt. 3]). The free group F admits therefore a basis $\{x_1, \dots, x_{m-1}, x_m\}$ such that U is the normal closure of the basis elements x_1, \dots, x_{m-1} and that $\eta(x_m)$ equals t . So \bar{U} is a free $\mathbb{F}_p(\text{gp}(t))$ -module of rank $m - 1$, as claimed.

We assert that the $\mathbb{F}_p(\text{gp}(t))$ -module \bar{R} can be generated by $m - 2$ elements. This is clear in case (i). In case (ii) it holds because a relator of the form $r = w^k$, with p dividing k , maps to $\bar{1} \in \bar{R}$. Indeed, since $r = w^k \in R \subset U$, the root w represents an element $\bar{w} \in F/U$ of finite order; as $F/U \xrightarrow{\sim} \mathbb{Z}$ is torsion-free, w belongs therefore to U and so w^p maps to $1 \in \bar{U} = U/(U' \cdot U^p)$.

So the $\mathbb{F}_p(\text{gp}(t))$ -module \bar{R} is again generated by $m - 2$ elements. In case (iii), one sees similarly that \bar{R} is generated by $m - 2$ elements. The module \bar{N} is therefore isomorphic to the quotient of a free $\mathbb{F}_p(\text{gp}(t))$ -module of rank $m - 1$ by a submodule generated by $m - 2$ elements, thus \bar{N} is an *infinite* elementary-abelian p -group and so the character χ represents a point outside of $\Sigma^1(G)$ by Lemma B3.21. Moreover, since $\mathbb{F}_p(\text{gp}(t))$ is a principal ideal domain, the module \bar{N} maps actually onto a copy of the free cyclic module $\mathbb{F}_p(\text{gp}(t))$, whence the wreath product $(\mathbb{Z}/p\mathbb{Z}) \wr C_\infty$ is a quotient of the group G . Note, though, that in case (iii) the abelianization of G can be finite; so one needs the extra hypothesis to justify the addendum.

So far we know that $\Sigma^1(G)$ contains no rank 1 point. Consider now a normal subgroup $M \subseteq G'' \cdot (G')^p$ and let $\pi: G \twoheadrightarrow Q = G/M$ be the canonical projection. This projection induces an isomorphism of spheres $\pi^*: S(Q) \xrightarrow{\sim} S(G)$. Under this isomorphism, a character $\bar{\chi}: Q \rightarrow \mathbb{R}$ corresponds to the character $\chi = \bar{\chi} \circ \pi$ of G . Now $\ker \bar{\chi} = (\ker \chi)/M = N/M$ and $M \subseteq G'' \cdot (G')^p \subseteq N' \cdot N^p$. So $\ker \bar{\chi}$ maps onto the infinite elementary p -group $N/(N' \cdot N^p)$ whence $[\bar{\chi}] \notin \Sigma^1(Q)$, again by Lemma B3.21.

The preceding paragraph shows that $\Sigma^1(Q)$ contains no rank 1 points. By Lemma B3.24 below these points are dense in $S(Q)$. As $\Sigma^1(G)$ is an open subset of $S(G)$ (by Theorem A3.3), the subset $\Sigma^1(Q)$ is therefore empty. \square

In the above proof the lemma given next has been quoted.

LEMMA B3.24 *The rank 1 points constitute a dense subset in the sphere $S(G)$ of a finitely generated group G .*

Proof. In view of the coordinate isomorphism (A1.3) it suffices to prove the claim for the unit sphere \mathbb{S}^{n-1} . Given $u \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$, there is a rational point $x \in \mathbb{Q}^n$ with $\|x\| \geq 1$ and $\|u - x\| \leq \varepsilon$. Then the distance from u to $x/\|x\|$ is at

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most ε ; indeed, the triangle with vertices u , $x/\|x\|$ and the origin o is isosceles and so the foot \hat{u} of the perpendicular from u to the line through o and x lies on the segment $[0, x/\|x\|]$, whence $d_2(u, x/\|x\|) \leq d_2(u, x) \leq \varepsilon$. \square

REMARKS B3.25 a) Proposition B3.23 is a refinement of [BNS87, Theorem 7.2] which, in turn, generalizes the main proposition in [BS81]. In both results, one views \bar{R} as a module over the principal ideal domain $\mathbb{F}_p(\text{gp}(t))$, an idea that can already be found in Section 2 of Baumslag's paper [Bau76].

b) The proposition implies that *every normal subgroup N with G/N infinite cyclic is infinitely generated whenever G is finitely presented with deficiency greater than 1*. This conclusion holds in far greater generality but the proof of this generalization relies on two results from the theory of L_2 -Betti numbers: firstly, the L^2 -Betti number $\beta_1(G)$ is 0 whenever G is a finitely presented group that admits a finitely generated normal subgroup N with an infinite quotient group G/N (see [Gab02, Thm. 6.8]). Secondly, the L^2 -Betti number $\beta_1(G)$ and the deficiency of a finitely presented group G are related by the inequality $\text{def}(G) \leq 1 + \beta_1(G)$ (see, e. g., [Hil97, Thm. 2]). It follows that *a finitely presented group with $\text{def}(G) > 1$ does not contain a finitely generated normal subgroup $N \neq \{1\}$ with infinite quotient G/N* .

We continue with some examples illustrating the use of Proposition B3.23.

EXAMPLES B3.26 a) *One relator groups G with at least 3 generators*. Then the deficiency is at least 2 and so the invariant $\Sigma^1(G/M)$ is empty for every quotient group $Q = G/M$ of G with $M \subseteq G'' \cdot (G')^p$; here p is a suitable prime number.

Well-known specimens of one-relator groups with deficiency at least 2 are non-abelian free groups and non-abelian, orientable surface groups. These later groups admit presentations of the form

$$\langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle \text{ with } g > 1. \quad (\text{B3.9})$$

They are special cases of discrete and co-compact groups of orientation-preserving automorphisms of the hyperbolic plane; these more general groups admit presentations of the form

$$\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_\ell \mid [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_\ell, c_1^{k_1}, \dots, c_\ell^{k_\ell} \rangle. \quad (\text{B3.10})$$

Here $k_i > 1$ for $i = 1, \dots, \ell$, and either $g > 1$, or $g = 1$ and $\ell > 0$. (See, e. g., the theorem on page 98 in [HKS71]). The deficiency of these groups is at least 1; as these presentations involve relators that are proper powers, Proposition B3.23 applies also to these more general groups. Note, however, that for $g = 1$, the prime p has to be chosen as a divisor of one of the exponents k_1, \dots, k_ℓ .

b) *Fundamental groups of connected, orientable and bounded 3-manifolds*. The deficiency of these groups is positive if the manifold M has a boundary component of positive genus and it is greater than 2 if one of its boundary components has genus greater than 1 (see, e. g., Lemma V.3 in [Jac80]).

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c) *One-relator groups with torsion.* Let G be a one relator group, say $G = \langle x_1, \dots, x_m \mid r \rangle$, and assume that $m \geq 2$ and that the defining relator r is a proper power of a non-empty word w . Then case (ii) of Proposition B3.23 is satisfied.

d) *Two generator two relator groups.* We finally consider groups with a presentation of the form

$$G = \langle x, y \mid r_1 = u^k, r_2 = w^\ell \rangle \text{ with } u \neq 1, w \neq 1 \text{ and } k > 1, \ell > 1.$$

The abelianisation of such a group can be finite; indeed, G can be an infinite dihedral group (and thus a finitely related metabelian group). We concentrate therefore on some special presentations where G_{ab} is infinite, namely

$$G = \langle x, y \mid r_1 = [x, y]^2, r_2 = y^\ell \rangle \text{ with } \ell > 1. \quad (\text{B3.11})$$

Let $\chi: G \rightarrow \mathbb{R}$ be the character given by $\chi(x) = 1$ and let N be the kernel of χ . Then N is the normal closure of y and the Reidemeister-Schreier procedure, applied with the Schreier transversal $\mathcal{T} = \{x^j \mid j \in \mathbb{Z}\}$, furnishes the presentation

$$N = \langle \{y_j\}_{j \in \mathbb{Z}} \mid (y_{j+1}y_j^{-1})^2 \text{ and } y_j^\ell \text{ for } j \in \mathbb{Z} \rangle. \quad (\text{B3.12})$$

of N . In this presentation the generator y_j , with $j\mathbb{Z}$, maps onto $x^j y x^{-j}$.

Assume first that ℓ is *even*. Then case (iii) of Proposition B3.23 holds and G_{ab} is infinite. The normal subgroup N maps obviously onto the abelian group

$$\bar{N} = \langle y_j \mid [y_j, y_k] = y_j^2 = 1 = \text{ for } j \in \mathbb{Z} \text{ and } (j, k) \in \mathbb{Z}^2 \rangle$$

which is isomorphic to the additive group of the group algebra $(\mathbb{Z}/2\mathbb{Z})\text{gp}(x)$, in accordance with the addendum to the proposition.

Assume now that ℓ is *odd*. Then the hypothesis of case (iii) is not satisfied; things improve, however, if one passes to a normal subgroup G_1 of index ℓ , namely the kernel of the homomorphism $\rho: G \rightarrow \mathbb{Z}/\ell\mathbb{Z}$ given by $\rho(x) = 0 + \ell\mathbb{Z}$ and $\rho(y) = 1 + \ell\mathbb{Z}$. The set $\mathcal{T} = \{1, y, \dots, y^{\ell-1}\}$ is a Schreier transversal for G_1 and the Reidemeister-Schreier procedure based on this transversal leads to the following presentation

$$\langle x_0, x_1, \dots, x_{\ell-1} \mid (x_0 \cdot x_1^{-1})^2, (x_1 \cdot x_2^{-1})^2, \dots, (x_{\ell-1} \cdot x_0^{-1})^2 \rangle. \quad (\text{B3.13})$$

for G_1 . This presentation satisfies the hypothesis for case (iii) of Proposition B3.23 and so $\Sigma^1(G_1)$ is empty; moreover, as the abelianization of G_1 has torsion-free rank 1, the groups G_1 maps onto the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr C_\infty$ by the addendum. The group G itself does not map onto a wreath product of the form $(\mathbb{Z}/p\mathbb{Z}) \wr C_\infty$ with p a divisor of ℓ ; in fact, under a homomorphism $\varphi: (\mathbb{Z}/p\mathbb{Z})C_\infty \rtimes C_\infty$ the commutator $[x, y]$ must map into $(\mathbb{Z}/p\mathbb{Z})C_\infty$, hence onto the 0 element and whence the image of φ is abelian. The invariant $\Sigma^1(G)$, however, is empty, just as is that of G_1 (use Proposition B1.11).

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B3.3b Groups having no fp extension without non-abelian free subgroups

In this section we have a new look at Proposition B3.20. To get the proper perspective, I quote a passage from the introduction to the recent article [BGdlH12] by M. G. Benli, R. Grigorchuk and P. de la Harpe.

The introduction to [BGdlH12] begins thus:

In the study of finiteness conditions on groups, the following kind of question is natural:

QUESTION 1.1 *Given a Property (\mathcal{P}) of groups, is any finitely generated group with (\mathcal{P}) a quotient of some finitely presented group with (\mathcal{P})?*

The answer can be positive for trivial reasons, for example when Property (\mathcal{P}) holds for free groups (such as exponential growth) or when Property (\mathcal{P}) implies finite presentation (such as nilpotency, or polynomial growth); [...]

*Here, we concentrate on a case with a negative answer; the goal of this article is to study examples and results concerning finitely generated amenable groups that do not have finitely presented amenable extensions. More precisely, amenable groups will often be groups of subexponential growth, and sometimes soluble groups; and non-amenable groups will usually have non-abelian free subgroups. Recall that an **extension** of a group G is a group E given together with an epimorphism $E \twoheadrightarrow G$.*

In the previous quotation amenable groups are mentioned; they are defined like this: a group G is *amenable* if there exists a finitely additive, left-invariant measure μ , defined on the set of all subsets of G , and such that $\mu(G) = 1$. Abelian and finite groups are amenable, and the class of amenable groups is closed under extensions and directed unions, and the formation of subgroups and quotient-groups. (For some further informations we refer the reader to section 2 of the survey [CS01] and the many references cited therein.)

Now to some examples illustrating the new look at Proposition B3.20.

EXAMPLES B3.27 a) Let G be finitely generated group that contains an infinite, locally finite normal subgroup N with infinite cyclic quotient G/N . Then $S(G)$ is a 0-dimensional sphere and $\Sigma^1(G)$ is empty by Lemma B3.21. The group G admits, of course, finitely presented extensions $E \twoheadrightarrow G$ of G , but according to Proposition B3.20 every such extension contains non-abelian free subgroups. (Concrete examples of such groups G are given in Examples B3.34.)

b) Elementary amenable groups are a sweeping generalization of the groups just discussed. By definition, the class EG of *elementary amenable groups* is the smallest class of groups that contains all finite and all abelian groups and is closed under extensions and directed unions. All soluble groups, but also all locally finite or locally nilpotent groups, are in EG. The class is also closed under the

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passages to subgroups and to quotient groups (see [Cho80] for further informations on elementary amenable groups).

A finitely generated, elementary amenable group G has no non-abelian free subgroup. If it admits a rank 1 character $\chi: G \rightarrow \mathbb{R}$ such that $[\chi]$ and $[-\chi]$ are both outside of $\Sigma^1(G)$, then Proposition B3.20 applies and guarantees that every finitely presented extension has a free subgroup of rank 2.

B3.3c Large finitely presented groups

Proposition B3.20, in conjunction with Proposition B3.23, allows one to prove that finitely presented groups with few relators contain non-abelian free subgroups, but by combining the addendum to Proposition B3.23 with a result of G. Baumslag, one can obtain a better conclusion and show that the groups are large. Here, following M. Gromov in [Gro82, p. 82, Thm. (B)], a group will be called *large* if it has a subgroup of finite index that maps onto a non-abelian free subgroup.

In the sequel, I shall reprove the mentioned result of G. Baumslag. Its proof is fairly tricky and based on ideas that go back to the paper [BP78] by B. Baumslag and S. J. Pride. I begin by explaining the gist of the proof with the help of a simple example.

EXAMPLE B3.28 Let G be a one-relator group with 3 generators, say $G = \langle a, b, t \mid r \rangle$. Assume r has exponent sum 0 with respect to t and let $\chi: G \rightarrow \mathbb{R}$ be the rank 1 character sending t to 1 and the other generators to 0. Let F be the free group on a, b , and t and let $\eta: F \rightarrow E$ denote the epimorphism involved in the presentation of G . Then r lies in the normal subgroup $R = \text{gp}_F(a, b)$ of F . We may and shall assume that r is a word in the conjugates of a and b by non-negative powers of t , say a word in the conjugates

$$a_0 = a, a_1 = tat^{-1}, \dots, a_m = t^m bt^{-m} \text{ and } b_0 = b, \dots, b_m = t^m bt^{-m}$$

and their inverses. Not all of the listed generators need actually occur in r . Put now $\ell = m + 1$ and $G_\ell = \text{gp}_G(a, b, t^\ell)$. The set $\mathcal{T} = \{1, t, \dots, t^m\}$ is then a Schreier transversal of G_ℓ in G , and the Reidemeister-Schreier procedure based on it leads to the generators

$$\begin{aligned} a_0 &\mapsto a, a_1 \mapsto tat^{-1}, \dots, a_m \mapsto t^{\ell-1}at^{-m}, \\ b_0 &\mapsto b, b_1 \mapsto tbt^{-1}, \dots, b_{\ell-1} \mapsto t^{\ell-1}bt^{1-\ell}, \quad \text{and} \quad T \mapsto t^\ell. \end{aligned}$$

The relators of G_ℓ are obtained by conjugating r by the elements of the transversal and rewriting the conjugates in terms of the generators listed above. If the last conjugate $t^{\ell-1}rt^{1-\ell}$ is rewritten, the generators occurring in it are among the words

$$\begin{aligned} a_m, \quad a_{m+1} &\mapsto Ta_0T^{-1}, \quad a_{m+2} \mapsto Ta_1T^{-1}, \dots, a_{2m} \mapsto Ta_{m-1}T^{-1}, \\ b_m, \quad b_{m+1} &\mapsto Tb_0T^{-1}, \quad b_{m+2} \mapsto Tb_1T^{-1}, \dots, b_{2m} \mapsto Tb_{m-1}T^{-1}. \end{aligned}$$

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A *first essential observation*, going back to the Baumslag-Pride paper [BP78], is now this: in the conjugate $t^m r t^{-m}$ the generator T occurs only in subwords of the form $T a_j T^{-1}$ with $0 \leq j \leq m-1$. This fact continues to be true for the other conjugates $r, t r t^{-1}, \dots, t^{m-1} r t^{1-m}$ of r .

Now to a *second, crucial idea* of B. Baumslag and S. J. Pride: let $M \triangleleft E_\ell$ be the normal closure of the elements

$$a_0, a_1, \dots, a_{m-1} \text{ and } b_0, b_1, \dots, b_{m-1}$$

and set $Q_\ell = G_\ell/M$. The observation implies then that the generator T does not occur in any relator of the presentation of Q_ℓ that is obtained from that of G_ℓ by adding the listed generators a_j and b_j as relators. Put differently, Q_ℓ is the free product of a subgroup K and the infinite cyclic group generated by T . Moreover, the character $\chi: G \rightarrow \mathbb{R}$ induces a character $\bar{\chi}: Q_\ell \rightarrow \mathbb{R}$ with K in its kernel and $\bar{\chi}(T) = \ell$.

The idea is now to show that the group $G_\ell = K \star \text{gp}(T)$ has a subgroup of finite index which maps onto a non-abelian free group. If K is the trivial group this is impossible. To exclude this and other unpleasant cases, B. Baumslag and S. J. Pride assume in [BP78] that the deficiency of the presentation defining E is at least 2 and deduce that K is a finitely presented group of deficiency at least 1 and so maps onto \mathbb{Z} .

Approach taken by G. Baumslag. In his lecture notes [Bau93], G. Baumslag shows that a finitely presented group is large whenever it maps onto the wreath product $W = \text{gp}(b) \wr C_\infty$ of a cyclic group of order some prime p and an infinite cyclic group. The details of G. Baumslag's proof are as follows:

assume G is a finitely presented group that admits an epimorphism $\rho: G \twoheadrightarrow W$ onto the wreath product $W = \text{gp}(b) \wr \text{gp}(s)$ of a finite cyclic group $\text{gp}(b)$ of order a prime number p by an infinite cyclic group generated by s . There exists then a finite set of generators $\{a^{(1)}, \dots, a^{(f)}, t\}$ of G with $\rho(a^{(1)}) = b$, with $a^{(2)}, \dots, a^{(f)}$ in the kernel of ρ and with $\rho(t) = s$. Let $\psi: W \rightarrow \mathbb{R}$ be the (unique) character that sends s to 1 and set $\chi = \psi \circ \rho$. Define F be the free group with basis $\{a^{(1)}, \dots, a^{(f)}, t\}$ and let $\eta: F \twoheadrightarrow G$ denote the obvious epimorphism. The kernel of η is then the normal closure of a finite set \mathcal{R} of relators (here one uses that G is finitely presentable and Lemma 8 in [Neu37]).

Set $R = \text{gp}_F(a^{(1)}, \dots, a^{(f)})$. Then \mathcal{R} is contained in R ; upon conjugating the elements of \mathcal{R} by suitable non-negative powers of t , there exists a positive integer m such that each relator $r \in \mathcal{R}$ can be written as a word in the generators $t^j a^{(i)} t^{-j}$ with $1 \leq i \leq f$ and $j \in \{0, 1, \dots, m\}$. Next, set $\ell = m+1$ and consider the subgroup $G_\ell = \text{gp}_G(t^\ell, a^{(1)}, \dots, a^{(f)})$ having index ℓ in G . Use the Schreier transversal $\mathcal{T} = \{1, t, \dots, t^m\}$ of G_ℓ in G to obtain a finite presentation of G_ℓ . Let now $M \triangleleft E_\ell$ be the normal closure of all the conjugates $t^j a^{(i)} t^{-j}$ with $j \in \{0, 1, \dots, m-1\}$ and $1 \leq i \leq f$. It then follows, as in the Example B3.28, that $Q_\ell = G_\ell/M$ is a free product $K \star \text{gp}(t^\ell)$ with

$$K = (\text{gp}(t^m a^{(1)} t^{-m}, \dots, t^m a^{(f)} t^{-m}) \cdot M) / M.$$

Notes

We claim that K maps onto a cyclic group of order p .

To justify this, we recall the definition of the projection $\rho: G \twoheadrightarrow W$; here W is the wreath product $\text{gp}(b) \wr \text{gp}(s)$. By definition, ρ sends the generators $a^{(2)}, \dots, a^{(f)}$ to $1 \in W$, then $a^{(1)}$ to the generator b and finally t to s . Let ρ_ℓ be the restriction of ρ to $G_\ell = \text{gp}_G(t^\ell, a^{(1)}, \dots, a^{(f)})$. The image of ρ_ℓ in W is the subgroup

$$W_\ell = \text{gp}(b, sbs^{-1}, \dots, s^mbs^{-m}) \wr \text{gp}(s^\ell) = (\mathbb{Z}/p\mathbb{Z}) \text{gp}(s) \rtimes \text{gp}(s^\ell).$$

The normal subgroup $M \triangleleft G_\ell$ is the normal closure of all the conjugates $t^j a^{(i)} t^{-j}$ with $j \in \{0, 1, \dots, m-1\}$ and $1 \leq i \leq f$. So $\rho_\ell(M)$ is the normal subgroup in W_ℓ generated by the group ring elements $1, t, \dots, t^{m-1}$. We conclude that ρ_ℓ induces a projection of $Q_\ell = G_\ell/M$ onto a wreath product of the form $\mathbb{Z}/p\mathbb{Z} \wr \text{gp}(s^\ell)$. Under this projection the factor K of $Q_\ell = K \star \text{gp}(t^\ell)$ maps onto the cyclic group $\mathbb{Z}/p\mathbb{Z}$, as asserted.

So far we know that G maps onto the free product $L = (\mathbb{Z}/p\mathbb{Z}) \star \text{gp}(s^\ell)$ of a cyclic group of order p and an infinite cyclic group. Consider now the projection $\pi_1: L \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ onto the first free factor. Its kernel is then a free group of rank $p \geq 2$ (this follows, e. g., by the Reidemeister-Schreier procedure). We have thus established the following result of G. Baumslag's:

THEOREM B3.29 ([BAU93, THM. IV.3.7]) *Let G be a finitely generated group that maps onto a wreath product $W = (\mathbb{Z}/p\mathbb{Z}) \wr C_\infty$ of a cyclic group of order a prime p and an infinite cyclic group. If G admits a finite presentation, it is large.*

B3.3d Examples of large finitely presented groups

We conclude section B3.3 with some applications of Theorem B3.29. This theorem presupposes that the finitely presented group E admits an epimorphism $\rho: G \twoheadrightarrow W$ onto the wreath product $W = (\mathbb{Z}/p\mathbb{Z}) \wr C_\infty$ of a cyclic group of order a prime p by an infinite cyclic group. This hypothesis implies that the sphere $S(G)$ is non-empty and also that $\Sigma^1(G)^c$ contains an antipodal pair of rank 1 points (use Lemma B3.21).

The statement of Proposition B3.23 lists three easily formulated assumptions that imply the hypothesis of Baumslag's theorem. The first two of these assumptions are illustrated by the examples given below.

1) Groups of deficiency at least 2. If G is a finitely presented group of deficiency at least 2 the sphere $S(G)$ has positive dimension and, for every rank 1 character χ and for every prime p , the kernel N of χ maps onto an infinite elementary p -group and hence onto a copy of the additive group of the group algebra $(\mathbb{Z}/p\mathbb{Z})C_\infty$ (by the addendum in the statement of Proposition B3.23). So Theorem B3.29 applies and shows that G is large, a conclusion first established by B. Baumslag and S. J. Pride in [BP78].

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2) *Groups of deficiency 1.* A group G of deficiency 1 need not be large, witnesses being the free abelian groups of rank 1 or 2 or, more generally, the soluble Baumslag-Solitar groups $BS(1, \ell) = \langle a, t \mid tat^{-1} = a^\ell \rangle$ with $\ell \in \mathbb{Z}$.

The situation is more complex for the non-soluble Baumslag-Solitar groups

$$G_{k,\ell} = BS(k, \ell) = \langle a, t \mid ta^k t^{-1} = a^\ell \rangle,$$

say with $k > 1$ and $\ell > 1$. If k and ℓ have a common factor $d > 0$ then $BS(k, \ell)$ maps obviously onto $(\mathbb{Z}/d\mathbb{Z}) \star \text{gp}(t)$ and so it is large. If, on the other hand, k and ℓ are relatively prime then $BS(k, \ell)$ is not large (see [EP84, Example 3.2]), in spite of the fact that $\Sigma^1(G_{k,\ell})$ is empty (use that Σ^1 of the metabelian top of $G_{k,\ell}$ is empty by Example A3.6) and $G_{k,\ell}$ contains non-abelian free subgroups.

The strategy of M. Edjvet and S. J. Pride is this: If G is a finitely generated large group it contains subgroups $M_1 \triangleleft H_1 \leq G$ so that H_1/M_1 is free of finite rank $r \geq 2$ and H_1 has finite index in G . Since the ranks of the finite index subgroups of H_1/M_1 are not bounded, G will have subgroups of finite index H whose abelianisation H_{ab} has arbitrary large torsion-free rank. A finitely generated group L is therefore not large if there exists an integer $b > 0$ so that the inequality $r_0(H_{\text{ab}}) < b$ holds for every subgroup H of finite index in L . Edjvet and Pride now show by a tricky argument that if $k > 1$ and $\ell > 1$ are relatively prime integers and $H < BS(k, \ell)$ is a subgroup of finite index, then $r_0(H_{\text{ab}}) < 2$; see [EP84, Ex. 3.3].

Another group of deficiency 1 that is not large is the one relator group

$$G = \langle a, t \mid tat^{-1} \cdot a \cdot ta^{-1}t^{-1} = a^2 \rangle \quad (\text{B3.14})$$

G. Baumslag detected in the late 1960s that all finite quotients of this group are cyclic (see [Bau69]). It follows easily that G is not large. The group G has, however, non-abelian free subgroups and $\Sigma^1(E)$ is empty. These facts can be established by adapting the proof of Theorem B3.1, to the given presentation of G . If one proceeds as in the first proof of Theorem B3.1 one discovers, with the help of the Freiheitssatz, that G is the HNN-extension $\langle B, t \mid ta_0 t^{-1} = a_1 \rangle$ with base group $B = \langle a_0, a_1 \mid a_1 a_0 a_1^{-1} = a_0^2 \rangle$. The associated subgroups are $S = \text{gp}(a_0)$ and $T = \text{gp}(a_1)$; as they are clearly distinct from B , the invariant $\Sigma^1(G)$ is empty by Proposition B3.7 whence G has a non-abelian free subgroups by Proposition B3.20.

The above examples should not leave the impression that groups of deficiency 1 are rarely large. First of all, the addendum in the statement of Proposition B3.23) shows that E maps onto a wreath product whenever it admits a presentations of deficiency 1 in which one relator is a proper power, whence such a group G will be large by Theorem B3.29. (This consequence is due to M. Gromov in [Gro82, p. 291. Thm. (B')] and to R. Stöhr in [Stö83], independently.

Examples of groups fulfilling the stated assumptions include two generator one-relator groups with torsion. Moreover, the investigations of J. O. Button, carried out in [But08], have brought to light that many groups with a deficiency 1 presentation in which no relator is a proper power are large, too.

B3.4 Applications to infinitely related groups

The consequence of Theorem B3.1 that will be discussed in this section is nothing but a contraposition of the theorem. For ease of reference we state it as

PROPOSITION B3.30 *Assume G is a finitely generated group which contains no non-abelian free subgroup and admits a rank 1 character $\chi: G \rightarrow \mathbb{R}$ so that both $[\chi]$ and $[-\chi]$ are points outside of $\Sigma^1(G)$. Then G is not of type FP_2 over any commutative ring and it does not admit a finite presentation.*

REMARKS B3.31 If one wants to apply Proposition B3.30 to a finitely generated group containing no free subgroups of rank 2, one has to find a rank 1 character $\chi: G \rightarrow \mathbb{R}$ with $\{[\chi], [-\chi]\} \subseteq \Sigma^1(G)^c$.

a) For some groups, finding such a character is easy: in Application 1 below the group G is locally finite by infinite cyclic and so the sphere $S(G)$ has only two points. In the situation of Proposition B3.23 the sphere $S(G)$ is often infinite and so there are infinitely many candidates. But, as the invariant $\Sigma^1(G)$ is empty, all choices lead to the desired result.

b) For other groups, the search for an appropriate character may be difficult. Here is a case in point. Let \tilde{G} be a one-relator group with two generators, say $\tilde{G} = \langle x, y \mid r \rangle$, and assume that the abelianisation of \tilde{G} is free abelian of rank 2. Consider the metabelian quotient $G = \tilde{G}/\tilde{G}''$. Then $S(G)$ is a circle containing infinitely many rank 1 points. It is thus not clear whether a character exists that allows one to infer that G is infinitely related and, if so, how it can be found. Fortunately, the answer to both questions can be found algorithmically; see Theorem B in [Str81b].

B3.4a How to prove that a finitely generated group is infinitely related?

Prior to listing some applications of Proposition B3.30, I would like to describe the historical context in which the proposition came into being.

In Note B3.35 I have mentioned that, in [Neu37], B. H. Neumann studies the question as to whether there are finitely generated, infinitely related groups and that he answers it in the affirmative, giving two kinds of justifications. To do so, Neumann constructs groups suited to his aim; they do not shed light on the related question which of the finitely generated groups discussed in a branch of group theory, say in the *Theory of Infinite Soluble Groups*, admit of a finite presentation.

This second question is addressed by P. Hall in his influential paper [Hal54]. He proves that every extension of two finitely presented groups is again so (Lemma 1 on p. 426) and deduces that every polycyclic group admits a finite presentation (Corollary on p. 426). Extensions provide one way of producing new groups from given groups, the wreath product furnishes another one. It turns out, however, that the wreath product $K \wr L$ of two finitely presented non-trivial groups K and

L is rarely finitely presentable; indeed, as shown by G. Baumslag in [Bau61], this happens if, and only if, L is finite (Baumslag's proof is lengthy; shorter proofs can be found in [Str84] (Theorem 4) and in [LR04] (Result 11.1.2).)

In proving the stated characterization, Baumslag makes use of free products; in a later paper [Bau71], he invokes the theory of free products with amalgamation to prove that a specific group is infinitely related. At the beginning of his survey [Bau74], he lists some classes of finitely generated soluble groups the members of which have been shown to be infinitely related and turns then to methods that permit one to justify the listed results. On page 66, he writes:

In general there seem to be two ways of proving that a given finitely generated group G is not finitely presented.

The first of these is a two-stage procedure. The first stage is to produce an explicit presentation of G in terms of a finite set of generators and an infinite set of defining relations. The second stage is to prove that no finite subset of these relations suffice to define G . (The fact that G is not finitely presented no matter which finite system of generators one may choose for G is the content of a theorem of B. H. Neumann [Neu37].) Both parts of this procedure can be difficult to effect; the second tends to be the more awkward and often invokes the use of generalized free products.

The second way to prove that a finitely generated group G is not finitely presented is to show that its multiplier $m(G)$ is not finitely generated, for the multiplier of a finitely presented group is always finitely generated. Actually it was an open question for a while whether conversely a finitely generated group with a finitely generated multiplier is finitely presented. This is false; there are even metabelian counter-examples [...].

The proof of Proposition B3.30 uses the first way described in the above quotation, but differs from it in two important respects. First of all, it relies on the theory of HNN-extensions and so presupposes that the abelianisation of the group G be infinite. Many groups that are known to be infinitely related do not satisfy this assumption and so cannot be shown to be infinitely related with the help of the proposition; but every finitely generated infinite soluble group admits an indicable subgroup of finite index. Secondly, no details of the presentation enter into the hypotheses of the proposition: one merely assumes the existence of a finite presentation and derives a structural property.

We now give some typical applications of Proposition B3.30. In these applications, the verification that the points $[\chi]$ and $[-\chi]$ lie outside of Σ^1 will often be based on Lemma B3.21.

B3.4b Application 1: locally finite by infinite cyclic groups

We begin with the special case of Lemma B3.21 where the normal subgroup M is reduced to the unit element. Proposition B3.30 and the lemma then yield

COROLLARY B3.32 *Assume G is a finitely generated extension of an infinite, locally finite group by an infinite cyclic group. Then $\Sigma^1(G)$ is empty and G does not admit a finite presentation.*

REMARKS B3.33 a) Corollary B3.32 is an easy consequence of Theorem B3.1, i. e., of Theorem A in [BS78] published in 1978. In view of this fact it is surprising that the corollary has only been detected in the last few years; in fact, the earliest reference I am aware of is Theorem D in the paper [BM09] by G. Baumslag and C. H. Miller III. The corollary is also part of Corollary C.4 in [BGdlH12].

b) The conclusion of Corollary B3.32 need not be true if G is an extension of a locally finite group by an abelian group of torsion-free rank greater than 1. Examples testifying to this are provided by a sequence of groups introduced by C. H. Houghton in [Hou79] on p. 257; these groups are finitely related (see [Bro87a]). Their invariants will be determined in section C1.2c.

c) The situation can also be different if the kernel N is an *infinite torsion group that is not locally finite*. Then the group G can be finitely related, but if so it must be an ascending HNN-extension over a finitely generated base group (by Theorem B3.1). Groups meeting these requirements are hard to come by, but they exist.

The first such group has been constructed by R. Grigorchuk in [Gri98]; it is an example of a *finitely presented amenable group that is not elementary amenable*.

Later A. Yu. Ol'shanskii and M. V. Sapir manufactured another group that is an extension of an infinite torsion group by a cyclic group (see [OS02]). It was the first example of a finitely presentable non-amenable group that contains no non-abelian free subgroups.

EXAMPLES B3.34 If one wants to apply Corollary B3.32 one has to find an infinite locally finite group N that admits of an automorphism $\alpha: N \xrightarrow{\sim} N$ of infinite order, all in such a way that N is finitely generated as a $\text{gp}(\alpha)$ -group. Here are some well-known groups of this kind.

a) N is the base group of the wreath product $B \wr C_\infty$ of a finite group B by an infinite cyclic group C_∞ .

b) N is the group of all permutation of \mathbb{Z} that fix all but finitely many elements and α is the automorphism induced by conjugation by the (infinitary) permutation $j \mapsto j + 1$. The split extension $G = N \rtimes \text{gp}(\alpha)$ is then generated by two elements, namely by α and the transposition τ of two adjacent elements of \mathbb{Z} , say 0 and 1.

c) Far more sophisticated examples have been constructed by B. H. Neumann in his remarkable note [Neu37]. Here is a brief description of these groups.

Let S be an infinite subset of the odd integers $n \geq 5$ and let P_S denote the unrestricted direct product

$$P_S = \prod_{n \in S} A_n.$$

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Let $B = B_S$ denote the subgroup of P_S that is generated by the following two sequences $\alpha = \alpha_S$ and $\tau = \tau_S$ of even cycles:

$$\alpha_S: n \mapsto a_n = (1 \mapsto 2 \mapsto 3 \mapsto 1), \text{ the remaining elements being fixed,} \quad (\text{B3.15})$$

$$\tau_S: n \mapsto t_n = (1, 2, 3, \dots, n-1, n). \quad (\text{B3.16})$$

Neumann proves that two such groups B_S and $B_{S'}$ are isomorphic if, and only if, $S = S'$, whence there are 2^{\aleph_0} pairwise non-isomorphic groups B_S . At the end of his note he states that B_S contains the restricted direct product $D = \text{Dr}_{n \in S} A_n$, that the quotient group B_S/D does not depend on S and is isomorphic to the semi-direct product of the alternating group on a countably infinite set by an infinite cyclic group. (A proof of the second claim may be found in [dlH00, pp.65+66]). It follows that the normal closure $N = \text{gp}_{B_S}(\alpha)$ of the element α_S is an infinite locally finite group, and thus B_S is *locally finite by infinite cyclic*, as asserted.

The fact that N is locally finite is pointed out in [BM09] (see Corollary 11). To verify this fact it suffices to show that, for every $\ell \geq 0$, the subgroup

$$M_\ell = \text{gp}(\alpha, \tau \cdot \alpha \cdot \tau^{-1}, \dots, \tau^\ell \alpha \tau^{-\ell})$$

is finite. To do so, fix ℓ and consider the images $\bar{M}_{\ell,n}$ of M_ℓ under the various projections $\pi_n: B \rightarrow A_n$. If $n \geq \ell + 3$, this image does not depend on n . It follows that for each index $n_0 \geq \ell + 3$ the kernel of π_{n_0} will consist of sequences $(g_n)_{n \in S}$ that are equal to 1 for every $n \geq n_0$. But if so, the kernel of π_{n_0} , and hence the group M_ℓ , are finite.

NOTE B3.35 On pages 125–127 of his paper, B. H. Neumann discusses the question *whether there are any groups which have a finite number of generators but which cannot be defined by a finite set of relations*. He asserts that such groups exist and offers two proofs. The first is direct and consists in giving appropriate examples (actually Neumann constructs 2^{\aleph_0} examples no two of which are isomorphic). The second relies on the fact that there are only countably many finitely presentable groups. So most of the previously described groups B_S must be infinitely related. But more is true: by Corollary B3.32 we know that all of them are infinitely related. This fact seems to have been pointed out first by G. Baumslag and C. H. Miller III in [BM09, Thm. C]. (The authors present two proofs, one of them is based on [BS78, Thm. A].)

B3.4c Application 2: Infinitely related generalized soluble groups

It turns out that the conclusion of Corollary B3.32 holds for a larger class of groups. Let \mathcal{P} be a property of groups and G a finitely generated group which is an extension of a normal subgroup N , that is locally \mathcal{P} , by an infinite cyclic group. *We look for an easily stated condition that forces G to be infinitely related.* The condition should imply that G cannot be an ascending HNN-extension with finitely generated base group B contained in N .

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If \mathcal{P} is the property of being finite a suitable requirement is that N do not have \mathcal{P} , that is to say, be infinite (see Corollary B3.32). In general, we shall require that N does not satisfy \mathcal{P} and that \mathcal{P} has the following special feature

$$\left. \begin{array}{l} \text{If } B \text{ has } \mathcal{P} \text{ and } B_\infty \text{ is the limit of an infinite sequence} \\ B \twoheadrightarrow B_1 \twoheadrightarrow B_2 \twoheadrightarrow \cdots \text{ of copies of } B \text{ then } B_\infty \text{ has } \mathcal{P} \end{array} \right\}. \quad (\text{B3.17})$$

The following corollary then holds:

COROLLARY B3.36 *Let G be a finitely group that is an extension of N by an infinite cyclic group. Assume \mathcal{P} has the special feature stated in (B3.17) and that N is locally \mathcal{P} , but not \mathcal{P} . Then G is infinitely related.*

Proof. Thanks to Proposition B3.30 it suffices to verify that G is not an ascending HNN-extension with finitely generated base group B contained in N . If it were, then B , being finitely generated, would have property \mathcal{P} . As N would then be a directed union of the form $\bigcup_{\ell \geq 0} t^\ell B t^{-\ell}$ with $t \in G$, it would satisfy \mathcal{P} by property (B3.17), contrary to the hypothesis on N . \square

To construct explicit examples covered by the corollary, one has to fix a property \mathcal{P} with the special feature (B3.17). As pointed out before, the property of being *finite* qualifies and examples based on it have been discussed in Examples B3.34. The property of being *nilpotent* has also the special feature. Corollary B3.36 thus guarantees that a finitely generated group G which is locally nilpotent by cyclic, but not nilpotent by cyclic, does not admit a finite presentation.

Here is a well-known specimen of such a group. Let A be the free abelian group with basis $\{e_j \mid j \in \mathbb{Z}\}$ and G the group generated by the automorphisms τ and α of A given by

$$\begin{aligned} \tau(e_i) &= e_{i+1} \text{ for } i \in \mathbb{Z}, \\ \alpha(e_0) &= e_0 + e_1 \text{ and } \alpha(e_j) = e_j \text{ for } j \in \mathbb{Z} \setminus \{0\}. \end{aligned}$$

For every $n \geq 0$ the subgroup $B_n = \text{gp}(\alpha, \tau \circ \alpha \circ \tau^{-1}, \dots, \tau^n \circ \alpha \circ \tau^{-n})$ is then nilpotent of class $n+1$ and so $N = \text{gp}_G(\alpha)$ is locally nilpotent, but not nilpotent.

The group G is a variation of the characteristically simple groups due to D. H. McLain (see [Rob96, pp. 361–362]) and is an example of an elementary amenable group; it is described by C. Chou in [Cho80] as Example 1 on page 402.

B3.4d Application 3: free soluble groups

In the early 1970s, G. Baumslag and V. N. Remeslennikov discovered that there are many more finitely generated metabelian groups with a finite presentation than one might have suspected. The borderline between the finitely related and the finitely generated but infinitely related, metabelian groups had therefore become blurred. For further progress, a good supply of simply described finitely generated metabelian groups that could be shown to be infinitely related was thus called for.

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With this goal in mind, G. Baumslag considers in [Bau76] the metabelian top G/G'' of a finitely presented group G whose deficiency is greater than 1. He proves that the multiplier $H_2(G/G'', \mathbb{Z})$ is infinitely generated whence G/G'' cannot be defined by finitely many relators. (This implication follows from Hopf's formula for $H_2(-, \mathbb{Z})$; see Formula (9.2) in [HS97, Chapt. VI]).

Baumslag reaches his conclusion in two steps. First, he verifies (with the technique used in the proof of Proposition B3.23) that G maps onto the wreath product $W = \mathbb{Z}/p\mathbb{Z} \wr C_\infty$ of a finite cyclic group with order a prime p by an infinite cyclic group C_∞ ; the prime can be prescribed arbitrarily. The wreath product W has an infinitely generated multiplier (see, e. g., [LR04, p. 241]). Consider now the 5-term sequence

$$H_2(G/G'', \mathbb{Z}) \rightarrow H_2(W, \mathbb{Z}) \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}Q} N_{ab} \rightarrow G_{ab} \rightarrow W_{ab} \rightarrow 0$$

associated to the extension $N \triangleleft G/G'' \twoheadrightarrow W$ (see, e. g., [HS97, Cor. 8.2]). Since the finitely generated metabelian group G/G'' satisfies the ascending chain condition on normal subgroups, N_{ab} is finitely generated as a $\mathbb{Z}Q$ -module. So the exactness of the above sequence at the term $H_2(W, \mathbb{Z})$ and the fact that $H_2(W, \mathbb{Z})$ is infinitely generated force $H_2(G/G'', \mathbb{Z})$ to be likewise infinitely generated.

The given argument is no longer sound if one replaces the metabelian top G/G'' of G by a soluble quotient that is not metabelian, say by G/G''' , for the quotient may then not satisfy the maximal condition on normal subgroups. Proposition B3.30, however, can step in. Here is a summary of the way towards this goal.

If one combines Proposition B3.30 with Proposition B3.23, one arrives at the following generalization of Baumslag's result:

COROLLARY B3.37 *Let p be a prime number and \mathcal{V} be a variety of groups that contains the metabelian variety $\mathcal{A}_p \cdot \mathcal{A}$, but is not the variety of all groups.*

Suppose G is a finitely presented group satisfying one of the assumptions (i), (ii) or (iii) listed in the statement of Proposition B3.23 and that G_{ab} is infinite. Then the canonical image $\bar{G} = G/\mathcal{V}(G)$ of G in \mathcal{V} is infinitely related.

NOTE B3.38 Corollary B3.37 is an amplification of Theorem B in [BS78]. It generalizes several earlier results, notably Šmel'kin's theorem that a free soluble group F/F^d of derived length $d \geq 2$ admits a finite presentation if, and only if it is cyclic ([Šme65]) and Baumslag's Theorem F in [Bau74] dealing with one-relator metabelian groups.

B3.5 Improved structure theorem for fg indicable groups

Theorem B3.1 asserts that every indicable group of type FP_2 can be written as an HNN-extension with a finitely generated base group B and finitely generated associated subgroups S and T ; moreover, the base group B can be required to be contained in the kernel N of a given rank 1 character $\chi: G \rightarrow \mathbb{R}$. If one looks at

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the proof of the theorem one notices that the hypothesis that G be of type FP_2 is only used towards the end and that the construction employed in the proof works for every finitely generated group. The conclusion in the case of a finitely group will be weaker than that of Theorem B3.1, but, as detected by Y. de Cornulier and L. Guyot, it has useful applications, notably in the context of *condensation groups*.

In this last part of Section B3, I explain how the idea of the proof of Theorem B3.1 can be twisted so as to yield the justification of a new structure theorem and compare then the new result with its predecessor, Theorem B3.1.

B3.5a Statement and proof of the theorem

Let G be a finitely generated indicable group and let $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ a rank 1 character. Choose an element $t \in G$ with $\chi(t) = 1$. Since G is finitely generated, there exists a finite subset \mathcal{A} in $N = \ker \chi$ such that $\mathcal{A} \cup \{t\}$ generates G . The kernel N will then be the normal closure of \mathcal{A} in G .

For $m > 0$, we next define a subgroup B_m of N and subgroups S_m , $T_m = tS_mt^{-1}$ of B_m , similarly as we did in the proof of Theorem B3.1. We set

$$B_m = \text{gp}(\{t^\ell at^{-\ell} \mid 0 \leq \ell \leq m \text{ and } a \in \mathcal{A}\}),$$

and then $S_m = \text{gp}(\{t^\ell at^{-\ell} \mid 0 \leq \ell < m \text{ and } a \in \mathcal{A}\})$ and $T_m = tS_mt^{-1}$. These subgroups enter in the construction of the HNN-extension

$$G_m = \langle B_m, y_m \mid y_m \cdot s \cdot y_m^{-1} = \tau_m(s) \text{ for } s \in S_m \rangle,$$

the isomorphism $\tau_m: S_m \xrightarrow{\sim} T_m$ being given by conjugation by t . There exist unique epimorphisms $\rho_m: G_m \twoheadrightarrow G_{m+1}$ and $\lambda_m: G_m \twoheadrightarrow G$ that extend the inclusions $B_m \hookrightarrow B_{m+1}$, respectively $B_m \hookrightarrow N$, and map y_m to y_{m+1} , respectively to t . These epimorphisms satisfy the commutativity relation $\lambda_m = \lambda_{m+1} \circ \rho_m$ for every $m \geq 1$; they induce therefore an epimorphism

$$\lambda_\infty: \text{colim}_{m \rightarrow \infty} G_m \twoheadrightarrow G.$$

Let K_m denote the kernel of the limiting map $\lambda_m: G_m \twoheadrightarrow G$ and consider the action of G_m on the Bass-Serre tree X_m associated to the HNN-extension G_m . Since λ_m is injective on the base group B_m , the action of K_m on X_m is free and so K_m is a free group. We claim that λ_∞ is an isomorphism. This amounts to show that every element $x \in K_0$ lies in the kernel of the composition $\pi_m = \rho_{m-1} \circ \cdots \circ \rho_0: G_0 \twoheadrightarrow G_m$ for m large enough. Since $K_0 \subseteq N = \ker \chi$, every $x \in K_0$ is a product of conjugates of elements $a \in \mathcal{A}$ by powers of t ; hence a conjugate of x , say $x' = t^n xt^{-n}$ with $n \geq 0$, will be mapped into B_m for all sufficiently large m . Since λ_m is injective when restricted to B_m and as $x' \in K_0 = \ker(\lambda_m \circ \pi_m)$ it follows that $\pi_m(x') = 1$ and hence $\pi_m(x) = 1$.

Suppose now that, for some $j \geq 1$, the kernel K_j is the normal closure of a finite set, say of \mathcal{F} . Since G is the colimit of the G_m there exists then an index $h \geq j$ such that the canonical image of \mathcal{F} under the canonical epimorphism $G_j \twoheadrightarrow G_h$ is

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trivial in G_h . The canonical isomorphism $(\lambda_j)_*: G_j/K_j \xrightarrow{\sim} G$ factors therefore through G_h and so the epimorphism $\lambda_h: G_h \twoheadrightarrow G$ is actually an isomorphism.

The previous reasoning constitutes a proof of

THEOREM B3.39 *Let G be a finitely generated group and $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ a rank 1 character. Choose $t \in G$ with $\chi(t) = 1$ and set $N = \ker \chi$. Then G is a colimit of a sequence of epimorphisms $\rho_m: G_m \twoheadrightarrow G_{m+1}$ of finitely generated groups $G_m = \text{gp}(B_m, y_m)$ with the following properties:*

- a) N is the ascending union $\bigcup_{m \geq 0} B_m$ of finitely generated subgroups B_m ,
- b) each B_m is generated by finitely generated subgroups S_m and $T_m = tS_m t^{-1}$,
- c) each G_m is an HNN-extension of the form

$$\langle B_m, y_m \mid y_m \cdot s \cdot y_m^{-1} = \tau_m(s) \text{ for } s \in S_m \rangle,$$

- d) each epimorphism $\rho_m: G_m \twoheadrightarrow G_{m+1}$ is induced by the inclusion $B_m \hookrightarrow B_{m+1}$ and the assignment $y_m \mapsto y_{m+1}$, and each epimorphism $\lambda_m: G_m \twoheadrightarrow G$ is induced by the inclusion $B_m \subset G$ and the assignment $y_m \mapsto t$.

Then one of the following two statements holds:

- (i) there is an index m_0 such that $\lambda_{m_0}: G_{m_0} \twoheadrightarrow G$ is an isomorphism,
- (ii) the kernel of every epimorphism $\lambda_m: G_m \twoheadrightarrow G$ is free of infinite rank and infinitely generated as a normal subgroup.

NOTE B3.40 Theorem B3.39 and its proof are taken from Section 6 in [BCGS12].

B3.5b Comparison of the new structure theorem with Theorem B3.1

Let G and $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ be as in the statement of Theorem B3.39. The group G is then a colimit of a sequence of epimorphisms

$$G_1 \xrightarrow{\rho_1} G_2 \xrightarrow{\rho_2} G_3 \xrightarrow{\rho_3} \cdots$$

of finitely generated groups. Choose a free group F of finite rank that maps onto G_0 , say $\pi: F \twoheadrightarrow G_0$, and set $R = \ker(\lambda_0 \circ \pi)$. Then R is the ascending union of the normal subgroups $R_m = \ker(\rho_{m-1} \circ \cdots \circ \rho_1 \circ \pi)$.

Suppose now that *statement (i) in Theorem B3.39 does not hold*. Then there are infinitely many indices m with $R_m \subsetneq R_{m+1}$, so R cannot be the normal closure of a finite set and thus the group $F/R \xrightarrow{\sim} G$ is infinitely related.

Next let \mathcal{R} be a finite subset of R and let S be the normal closure of \mathcal{R} in F . Then there exists an index m_* so that $S \subseteq R_{m_*}$. Set $H = F/S$ and let $\pi_*: H = F/S \twoheadrightarrow F/R \xrightarrow{\sim} G$ be the obvious epimorphism; its kernel M is equal

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to R/S . The extensions $M \triangleleft H \twoheadrightarrow G$ and $K_{m*} \triangleleft G_{m*} \twoheadrightarrow G$ then fit into the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M = R/S & \longrightarrow & H = F/S & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow \text{can} & & \downarrow \text{can} & & \downarrow = \\ 0 & \longrightarrow & K_m = R/R_{m*} & \longrightarrow & G_m = F/R_{m*} & \longrightarrow & G \longrightarrow 0. \end{array} \quad (\text{B3.18})$$

Its rows are exact. Let K be a commutative ring (with $1 \neq 0$). The left vertical epimorphism induces then an epimorphism $K \otimes M_{ab} \twoheadrightarrow K \otimes (K_m)_{ab}$.

By statement (ii) in Theorem B3.39 the kernel K_m is a infinitely generated free group; its abelianisation is therefore a non-trivial free abelian group and so the K -module $K \otimes M_{ab}$ is non-zero. In view of Lemmata A5.9 and A5.10 this conclusion implies that G is not of type FP_2 over the ring K .

So far we have shown that the group G is infinitely related and that it is not of type FP_2 . These findings have previously been obtained from a contraposition of Theorem B3.1; see Proposition B3.30. But this assumption that statement (i) does not hold has further consequences: it shows that G is the colimit of a sequence of groups G_m having the property that each group G_m contains a normal subgroup K_m which is free of infinite rank. This later property implies that G_m has continuously many normal subgroups (by a basic result in the variety of groups, proved independently by Adian, Olshanskii and Vaughan-Lee in [Adj70], [Ol'70] and [VL70]) and that G_m a condensation point in the space of marked groups (by [BCGS12, Cor. 5.4]). It then follows that the colimit G is a condensation point in the space of marked groups.

Here is a summary of the insights obtained in the above:

COROLLARY B3.41 *Let G be a finitely generated group that admits a rank 1 character $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ such that G is not an HNN-extension with finitely generated base group B contained in $\ker \chi$ and finitely generated associated groups.*

Then G is not of type FP_2 over a non-zero commutative ring and it is a condensation point in the space of marked groups.

We close with a word on the verification that a rank 1 character does not satisfy statement (i) in Theorem B3.39. Let $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ be a rank 1 character with kernel N and pick $t \in \chi^{-1}(\{1\})$. Assume χ and $-\chi$ represent points outside of $\Sigma^1(G)$ and consider one of the approximating HNN-extensions G_m constructed in the proof of B3.39. Then the characters $\pm\chi \circ \lambda_m$ represent points outside of $\Sigma^1(G_m)$, so G_m cannot be an ascending HNN-extension (see Corollary B1.8) and thus G_m contains a non-abelian free subgroup. If G contains no such subgroup then λ_m cannot be an isomorphism; as this argument is valid for every index m , we have proved that statement (i) does not hold.

The hypothesis that G contain no non-abelian free subgroup is a familiar one in the applications of the Σ -theory; see, e. g., Theorem A5.1 or Proposition B3.30. Recently weaker substitutes have been found. They are listed in Proposition 6.6

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in [BCGS12] and lead to interesting applications; see Examples 6.7 and 6.11 in the cited paper.

B4 Computation of Σ^1 for one relator groups

In this section, we calculate the invariant of a group with m generators and a single defining relator r . The crucial case is that where $m = 2$ and r is a cyclically reduced word which involves both generators. In fact, if $m = 1$, the group G is cyclic and $\Sigma^1(G) = S(G)$; if $m > 2$ the invariant is empty by part (i) of Proposition B3.23; if, finally, $m = 2$ and r is a power of one of the generators, say if $r = a^\ell$, the group G is cyclic if $|\ell| = 1$ and a non-trivial free product otherwise. In the first case, $\Sigma^1(G) = S(G)$ consists of two points, in the other case it is empty (see Example A2.5 a) for the first case and Example 3 in section A2.1a or part (ii) of Proposition B3.23 for the second one).

The determination of the invariant in the remaining case is more challenging. It is due to Ken Brown ([Bro87b]) and described in

THEOREM B4.1 *Let G be a group given by a presentation $\eta_*: \langle a, b, | r \rangle$ where $r = s_1 \cdots s_k$ is a cyclically reduced, non-empty word involving both generators. Then a non-zero character $\chi: G \rightarrow \mathbb{R}$ represents a point of $\Sigma^1(G)$ if, and only if, the sequence*

$$f_r(\chi) = (\chi(s_1), \chi(s_1 s_2), \dots, \chi(s_1 \cdots s_k)) \quad (\text{B4.1})$$

satisfies the following condition:

$$\left. \begin{array}{l} \text{if one of } \chi(a) \text{ and } \chi(b) \text{ is zero, } f_r(\chi) \text{ assumes its minimum twice,} \\ \text{otherwise } f_r(\chi) \text{ assumes its minimum once.} \end{array} \right\} \quad (\text{B4.2})$$

REMARKS B4.2 a) Condition (B4.2) is the defining property of the set $\psi(\{r\})$ (see Definitions B2.2 and B2.6). So the conclusion of Theorem B4.1 can be restated by saying that the subset $\psi(\{r\})$ coincides with $\Sigma^1(G)$.

b) Theorem B4.1 will be obtained by combining four ingredients: the algebraic version of the Σ^1 -criterion embodied in the subset $\psi(\{r\}) \subseteq \Sigma^1(G)$, the standard method of analyzing one-relator groups due to W. Magnus [Mag30], Proposition B3.7 and the openness of $\Sigma^1(G)$.

B4.1 Proof of Theorem B4.1

If the sequence $f_r(\chi)$ satisfies condition (B4.2) then $[\chi]$ belongs to the subset $\psi(\{r\})$ (see Definition B2.6) and thus lies in $\Sigma^1(G)$ by Proposition B2.5. For the proof of the converse, three cases will be distinguished. We shall first treat the case where χ vanishes on one of the generators, then the case of a character that is non-zero on both generators but has rank 1. Finally, characters of rank 2 will be considered.

Notes

B4.1a Case 1

If χ vanishes on one of the generators, it has rank 1 and four, very similar subcases arise; it will suffice to analyze the situation of the character $\chi: G \twoheadrightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ with $\chi(a) = 1$ and $\chi(b) = 0$. Let F denote the free group on the generators $t = a$ and b , let $\eta_*: F \twoheadrightarrow G$ be the epimorphism induced by $\eta: \{t, b\} \rightarrow G$. Set $N = \ker \chi$ and $U = \ker(\chi \circ \eta_*) = \eta_*^{-1}(N)$.

The normal subgroup U is freely generated by the conjugates $b_j = t^j b t^{-j}$ of b with $j \in \mathbb{Z}$. The relator is a product of conjugates of b and of b^{-1} and can thus be written as a cyclically reduced word w in the letters b_j and b_j^{-1} ; let μ be the smallest and M the largest of the indices that occur when r is written as a word w in these letters; since r involves both generators, $\mu < M$. Define subgroups B , S and T by setting

$$B = \text{gp}(\{\eta_*(b_j) \mid \mu \leq j \leq M\}), \quad S = \text{gp}(\{\eta_*(b_j) \mid \mu \leq j < M\})$$

and $T = \text{gp}(\{\eta_*(b_j) \mid \mu < j \leq M\}) = \mu(S)$. By the Freiheitssatz (see, e. g., [LS01, p. 198]), the groups S and T are free on the displayed generators, whence G is an HNN-extension with base group B , associated subgroups S , T and stable letter t . The hypothesis that $[\chi]$ lies in $\Sigma^1(G)$ and Proposition B3.7 allow one to deduce that the base group B coincides with the associated subgroup T .

By going back to the definition of B and T one sees that the generator b_μ can be omitted from the given set of generators of B . Thus there exists a word v in the generators $b_{\mu+1}, \dots, b_M$ such that b_μ and v define the same element in B . But B , by the proof of the Freiheitssatz, is a group having $\mathcal{B} = \{b_\mu, \dots, b_M\}$ as its set of generators and a single defining relator w ; here w denotes a word obtained by rewriting the original relator r as a (cyclically reduced) word in the generators b_j . So B can be written as quotient Φ/R of the free group on \mathcal{B} modulo the normal subgroup R generated by w . By the previous paragraph B can also be written as Φ/R_1 where $R_1 \triangleleft \Phi$ denotes the normal subgroup generated by $w_1 = b_\mu \cdot v^{-1}$. Since $R_1 \subseteq R$ the identity on Φ induces an epimorphism $\rho: \Phi/R_1 \twoheadrightarrow \Phi/R$; as the groups Φ/R_1 and Φ/R are both isomorphic to the finitely generated free group T and free groups of finite rank are hopfian (see, e. g., [MKS04, Sec. 2.4, Thm. 2.13]), the epimorphism ρ is injective, whence R_1 equals R .

At this point, we invoke another theorem of W. Magnus', the *Conjugacy Theorem for Groups with One Defining Relator* (see Theorem 4.11 in [MKS04, p. 261]). It guarantees that the relator $w_1 = b_\mu \cdot v^{-1}$ is a cyclic permutation of w or of w^{-1} . But if so, the generator b_μ occurs only once in w , either as b_μ or b_μ^{-1} . This fact can be restated in terms of the original relator r by saying that the minimum of the sequence $f_r(\chi)$ occurs twice, once before an occurrence of b^\pm and once afterwards. (If the relators starts out with b^\pm and the minimum is 0, the preceding conclusion is to be interpreted as saying that the minimum occurs after the first letter and at the end of the relator.) Thus $f_r(\chi)$ satisfies condition (B4.2).

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B4.1b Case 2

In Case 2, the character $\chi: G \rightarrow \mathbb{R}$ has rank 1 and takes non-zero values on both generators. Its image can be assumed to be \mathbb{Z} ; set $p = \chi(a)$ and $q = \chi(b)$. Our aim is to reduce this set-up to that considered in Case 1. To do so, we adjoin to G a p -th root of a and a q -th root of b , ending up with a group \tilde{G} with generating set $\{x, y\}$, a character $\tilde{\chi}: \tilde{G} \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R}$ and a defining relator \tilde{r} . We then verify that $[\tilde{\chi}] \in \Sigma^1(\tilde{G})$, introduce a new generating system $\{x, z = yx\}$, express the relator \tilde{r} in terms of the new generators, reduce it and arrive thus at a relator of the type studied in Case 1. A comparison with Case 1 will then disclose that the sequence $f_{\tilde{r}}(\tilde{\chi})$ assumes its minimum only once and to conclude that the original sequence $f_r(\chi)$ has this property, too.

The details of the outlined verification are as follows. By replacing, if need be, the defining relator r with a cyclic permutation of itself, one can assume that r has the form

$$r = a^{e_1} b^{f_1} \cdot a^{e_2} b^{f_2} \cdots a^{e_\ell} b^{f_\ell} \quad (\text{B4.3})$$

where each of the exponents e_j, f_j is non-zero. Adjoin a p -th root x of a to G , obtaining a group with the presentation

$$G_1 = \langle a, b, x, \mid r(a, b), a \cdot x^{-p} \rangle \stackrel{\sim}{\leftarrow} \langle x, b \mid r(x^a, b) \rangle.$$

Then adjoin a q -th root y of b to G_1 , ending up with the group

$$\tilde{G} = \langle x, y \mid \tilde{r}(x, y) \rangle \quad \text{with} \quad \tilde{r}(x, y) = r(x^a, y^b) = x^{pe_1} \cdot y^{qf_1} \cdots x^{pe_\ell} \cdot y^{qf_\ell}. \quad (\text{B4.4})$$

The character χ of G extends uniquely to a character $\tilde{\chi}$ of \tilde{G} ; its image is \mathbb{Z} . By hypothesis, $[\chi]$ belongs to $\Sigma^1(G)$ and χ is non-zero on a as well as on b . Proposition B1.15 and the facts that the invariants of the infinite cyclic groups $\text{gp}(x)$ and $\text{gp}(y)$ coincide with the spheres $S(\text{gp}(x))$ and $S(\text{gp}(y))$, respectively, thus allow one to deduce that $\tilde{\chi}$ represents a point of $\Sigma^1(\tilde{G})$.

The function $f_{\tilde{r}}(\tilde{\chi})$ arises from the function $f_r(\chi)$ by “affine interpolation”. If $f_{\tilde{r}}(\tilde{\chi})$ assumes its minimum exactly once so will therefore the function $f_r(\chi)$. To justify the claim in Case 2, it suffices therefore to establish

LEMMA B4.3 *Assume L is a one-relator group with generators x, y and defining relator of the form*

$$u = x^{m_1} y^{n_1} \cdots x^{m_\ell} y^{n_\ell} \quad (\text{B4.5})$$

where $\ell > 0$ and all exponents m_i and n_i are non-zero. Let $\psi: L \rightarrow \mathbb{R}$ be the character that sends x and y to 1. If $[\psi] \in \Sigma^1(L)$ then the function $f_u(\psi)$ assumes its minimum only once.

Proof. By replacing u with a suitable cyclic conjugate and then, if need be, with the inverse of this conjugate, we can arrange that the minimum of the function $f_u(\psi)$ is 0 and that the new relator begins with x and ends with y^{-1} . We thus assume that $m_1 > 0$ and $n_\ell < 0$ and have to show that the ψ -value of no proper initial segment $s_1 \cdots s_h$ of u is 0.

Notes

To arrive at this goal, we introduce a new basis $\{x, z\}$ in the free group on $\{x, y\}$. Put $z = y \cdot x^{-1}$. Then ψ vanishes on z and $y = z \cdot x$. Upon substituting $z \cdot x$ for y in the given relator u one arrives at the new relator

$$\hat{u} = x^{m_1}(z \cdot x)^{n_1} \dots x^{m_\ell}(z \cdot x)^{n_\ell}. \quad (\text{B4.6})$$

This relator is not freely reduced and so we have to determine the relationship between \hat{u} and the word \hat{u}_{red} that is obtained from \hat{u} by free reduction. To see where the crux lies, we first look at

EXAMPLE B4.4 Consider the relator $u = xYxxYxYxyXYXyxyXYY$ where X, Y denote the inverses of x and y . The graph of the sequence $f_u(\psi)$ looks then as depicted in Figure B.10.

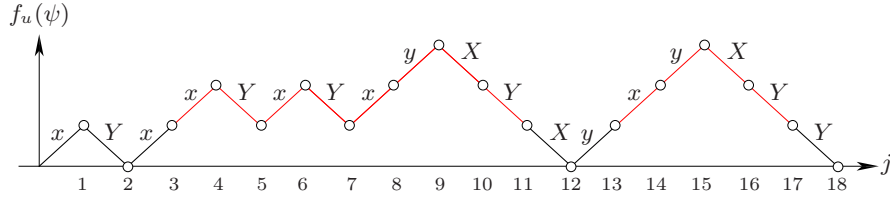


Figure B.10: Graph of the function $f_u(\psi)$

Now replace y and Y by zx, XZ , respectively; there results the word \hat{u} . This word is not reduced; if one reduces it, one obtains the word \hat{u}_{red} . Its graph is shown in Figure B.11.

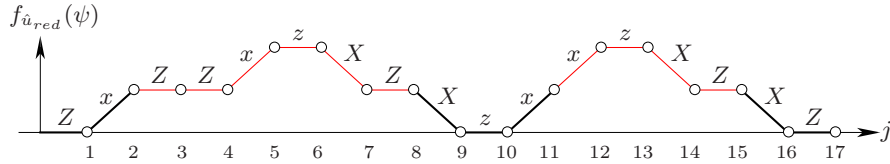


Figure B.11: Graph of the function $f_{\hat{u}_{red}}(\psi)$

A comparison of the two graphs leads to the following observations. The domain $\{1, 2, \dots, 18\}$ of the function $f_u(\psi)$ has five consecutive subintervals of maximal lengths; these subintervals are of two types: those of the first type are characterized by the fact that the function $f_u(\psi)$ assumes on them only the values 0 and 1 whereas $f_u(\psi)$ assumes at least once the value 2 on the intervals of the other type. The graphs of the function on these subintervals is drawn alternatively in black and in red. Let v_1, w_1, v_2, w_2 and v_3 be the subwords that give rise to these intervals. More precisely, let v_1 be the longest initial segment of u on which

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the function takes only the values 0 and 1. Next, let $v_1 w_1$ be the longest initial segment such that $f_u(\psi)$ takes on the second part only values in $\{1, 2, \dots\}$ and define the subwords v_2 , w_2 and v_3 similarly.

If the occurrences of the generator y in these subwords are replaced by zx , one obtains the words $\hat{v}_1, \hat{w}_1, \dots, \hat{v}_3$; they need not be reduced. Let $(\hat{v}_1)_{red}, (\hat{w}_1)_{red}, \dots, (\hat{v}_3)_{red}$ be their reductions. The crucial observations are now these: the word v_1 has the form $v_i = x(Yx)^k$ with $k \geq 0$ and $\hat{v}_i = x(XZx)^k$, whence $(\hat{v}_i)_{red} = Z^k x$. The word v_2 has the form $v = (Xy)^k$ with $k > 0$ and $\hat{v}_{red} = Xz^k x$. For future use, we note that the word $v' = (Yx)^k$ leads to $\hat{v}'_{red} = XZ^k x$. The word v_3 , finally, has the form $v_\ell = (Yx)^k Y$ with $k \geq 0$ and $(\hat{v}_\ell)_{red} = XZ^{k+1}$. Consider now the word $w = w_j$ with $j \in \{1, 2\}$. Then $f_w(\psi)$ is nowhere negative and has the value 0 on w . It follows that the functions $f_{\hat{w}}(\psi)$ and $f_{\hat{w}_{red}}(\psi)$ are nowhere negative, whence *the word \hat{w}_{red} cannot start with x^{-1} and cannot end in x .*

Now to the upshot of these observations. The word $(\hat{v}_1)_{red} \cdot (\hat{w}_1)_{red} \cdots (\hat{v}_3)_{red}$ has the form

$$Z^{k_i} x \cdot (\hat{w}_1)_{red} \cdot XZ^{\pm k} x \cdot (\hat{w}_2)_{red} \cdot XZ^{k_\ell+1}$$

with $k_i \geq 0$, $k > 0$ and $k_f \geq 0$. By the statement in italics, this product is therefore reduced as written; actually, it is even cyclically reduced. The function $f_u(\psi)$ assumes the minimum 0 in the first subinterval, in the third subinterval and in the last one and the transformed function $f_{\hat{u}_{red}}(\psi)$ has the analogous property.

We are now ready to deal with an arbitrary relator u of length at least 2. We get first rid of the exceptional case where $u = (xY)^k$ for some $k > 0$. Then $f_u(\psi)$ assumes k times the value 0. Since $\hat{u}_{red} = Z^k$ the function $f_{\hat{u}_{red}}(\psi)$ assumes its minimum also k times. In view of Case 1, the assumption that $[\psi] \in \Sigma^1(L)$ implies therefore that $k = 1$, or alternatively, that $f_u(\psi)$ assumes its minimum only once.

Suppose now that u is not of the form $(xY)^k$ and subdivide the domain $\{1, 2, \dots, m\}$ of $f_u(\psi)$ into subintervals v_1, w_1, \dots, v_ℓ as explained in the example. Three cases arise. If $v_1 = x(Yx)^{k_1}$ and $k_1 > 0$ the reduced word \hat{u}_{red} starts with $Z^{k_1} x$ and ends in $XZ^{k_\ell+1}$, whence the function $f_{\hat{u}_{red}}(\psi)$ assumes its minimum at least $k_1 + (k_\ell + 2) \geq 3$ times. Case 1 then allows one to conclude that $[\chi] \notin \Sigma^1(L)^c$. One sees similarly that $[\chi] \notin \Sigma^1(L)^c$ whenever $k_\ell > 0$.

In the third and last case, one has $v_1 = x$ and $v_\ell = Y$. If $\ell = 2$ then $u = v_1 w_1 v_\ell$ and so $f_u(\psi)$ assumes its minimum only once; if however, $\ell > 2$ the subword $v_1 \cdot w_1 \cdot v_2 = x \cdot w_1 \cdot (Xy)^{\pm k_2}$ of u will force the function $f_u(\psi)$ to have at least 2 absolute minima. The transformed word \hat{u}_{red} will then start with $x \cdot (\hat{w}_1)_{red} \cdot Xz^{\pm k_2} x$; since $k_2 > 0$ the function $f_{\hat{u}_{red}}(\psi)$ assumes therefore the values 0 at least $(1 + k_2) + 2 \geq 4$ times and so $[\chi] \notin \Sigma^1(L)^c$. This completes the verification of Lemma B4.3 and establishes Case 2 in the proof of Theorem B4.1. \square

B4.1c Case 3

We want to show that every rank 2 point of $\Sigma^1(G)$ is in the subset $\psi(\{r\})$ defined by condition (B4.2). The proof of this assertion is fairly easy and is best understood if interpreted in the geometric set-up described in section B2.2a.

Notes

Rank 2 points can only occur if G_{ab} is free abelian of rank 2. Let $\vartheta: G \rightarrow \mathbb{Z}^2$ be the epimorphism of G that sends the couple (a, b) to the standard basis $((1, 0), (0, 1))$ of the Euclidean plane \mathbb{R}^2 equipped with the usual scalar product. Then ϑ induces an isomorphism

$$\sigma(\vartheta): \mathbb{S}^1 \xrightarrow{\sim} S(G), \quad u \mapsto [g \mapsto \langle u, \vartheta(g) \rangle]$$

of circles (see section A1.1d). The relator $r = s_1 s_2 \cdots s_k$ of G gives rise to a sequence

$$(\vartheta(s_1), \vartheta(s_1) + \vartheta(s_2), \dots, \vartheta(s_1) + \cdots + \vartheta(s_k)) \quad (\text{B4.7})$$

of lattice points. The points of this sequence are the vertices of a path \bar{p} in the Cayley graph $\Gamma(\mathbb{Z}^2, \{a, b\})$ that ends in the origin; we shall think of \bar{p} as being a loop, starting at the origin and also ending there.

The determination of the subset $\psi(\{r\})$ amounts now to this. One seeks unit vectors $u = (u_1, u_2) \in \mathbb{S}^1$ for which the function $h_u: x \mapsto \langle u, x \rangle$ assumes its minimum at most twice along the vertices of the path \bar{p} , the vertices being counted with multiplicity. As an aid in finding these unit vectors, construct the convex hull of the set of vertices $v_j = \vartheta(s_1 \cdots s_j)$ of \bar{p} . The boundary \mathcal{C} of this hull is a closed, convex polygon whose vertices form a subset of the vertices of the path \bar{p} . Call a vertex v of \mathcal{C} *simple* if it equals v_j for exactly one j . Since the relator r has at least length 4, the polygon \mathcal{C} always contains horizontal edges at both the top and the bottom and a vertical edge on either side. Let e be one of these four edges. If e contains exactly two vertices, call it *special*. (Note that a special edge is necessarily of length 1 and that its two vertices are simple.)

The proof of Case 3 can now be completed like this. Each edge e of the polygon \mathcal{C} determines a unit vector $u_e \in \mathbb{S}^1$ that is orthogonal to the line ℓ_e carrying the edge e and has the property that the function $v_j \mapsto \langle u, v_j \rangle$ assumes its minimum in the vertices lying on e . This unit vector u_e has rank 1. A unit vector u of rank 2 lies therefore in the complement of the set $\{u_e \mid e \text{ edge of } \mathcal{C}\}$, and thus belongs to an open arc α bounded by endpoints u_e and $u_{e'}$, say. Let v_j be the common end point of the edges e and e' . If v_j is simple, the arc α is contained in $\psi(\{r\})$ and hence in $\Sigma^1(G)$; otherwise, $\alpha \cap \Sigma^1(G)$ contains no rank 1 point by Case 2 and hence no rank 2 points either, rank 1 points being dense in α (see Lemma B3.24).

REMARKS B4.5 a) The above proof of Brown's Theorem B4.1 is close to that given in [Bro87b, Section 4], the main differences being more detailed verifications and the fact that references to *HNN-valuations* have been replaced by references to the characterization of rank 1 points in terms of ascending HNN-extension with finitely generated base group (Proposition A3.4) and references to Proposition B2.5. In [BR88, Section 7], Bieri and Renz give a different proof of Theorem B4.1.

b) The core of Brown's result can be stated without using the invariant Σ^1 : given an epimorphism $\chi: G \rightarrow \mathbb{Z}$ and $t \in G$ with $\chi(t) = 1$, the sequence of values of χ on the initial segments $s_1 \cdots s_j$ of the relator $r = s_1 \cdots s_\ell$ allows one to decide whether G is an ascending HNN-extension $\langle B, t \mid B \subseteq t B t^{-1} \rangle$ with finitely generated base group B .

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c) Brown's Theorem yields an algorithm for deciding whether the kernel of a rank 1 character $\chi: G \rightarrow \mathbb{R}$ is finitely generated. This algorithm has found striking applications in 3-manifold theory; see [Dun01], [But05] and [DT06].

d) Let $G = \langle a, b \mid r \rangle$ be a one relator group and $\chi: G \rightarrow \mathbb{R}$ a rank 1 character with kernel N . Pull χ back to a character $\tilde{\chi}: F \rightarrow \mathbb{R}$ of the free group F on a, b . Then F_{ab} admits a basis (\bar{s}, \bar{t}) such that $\ker \tilde{\chi} \cdot F'$ is generated by $s \cdot F'$. Now every automorphism of F_{ab} is induced by an automorphism of F (see, e. g., [LS01, Chapt. I, Prop. 4.4]); so the group F admits an ordered basis (s, t) which projects to the previously chosen basis (\bar{s}, \bar{t}) of F_{ab} . It follows that G has a one relator presentation with generators s, t such that the kernel N of χ is the normal closure of the generator s .

Assume now that $[\chi] \in \Sigma^1(G)$. The analysis in Case 1 of the proof of Theorem B4.1, but with generators s, t instead of a, b , then shows that G is an ascending HNN-extension whose base group is free (of finite rank). Moreover, if both $[\chi]$ and $-[\chi]$ lie in $\Sigma^1(G)$ the kernel N of χ is free. The preceding reasoning implies that every finitely generated kernel of a rank 1 character of a one relator group is free. As shown by R. Bieri in [Bie07, Cor. B], this conclusion continues to be valid if G is a finitely presented group of deficiency 1.

B4.2 Some examples

The following two examples have different objectives: the first illustrates the actual computation of Σ^1 by means of Brown's algorithm; the second one reinterprets a result of G. Baumslag in the frame work of the theory of the invariants Sigma.

EXAMPLE B4.6 Figure B.12 shows diagrams of knots of types 8_{20} and 8_{21} , as listed in **KnotInfo**⁶; let G_{20} and G_{21} denote the corresponding groups. J. Weeks' program **SnapPea**⁷ is able to find one-relator presentations for them. For G_{20} one gets generators a, b and the defining relator

$$r_{20} = a^2 \cdot b^2 \cdot a \cdot b^{-2} \cdot a^{-1} \cdot b \cdot a^{-1} \cdot b^{-2} a \cdot b^2 \quad (\text{B4.8})$$

The exponent sums (σ_a, σ_b) of r_{20} are $(2, 1)$; so G_{20} admits a rank 1 character χ with $\chi(a) = 1$ and $\chi(b) = -2$. The sequence $f_{r_{20}}(\chi)$ reads thus

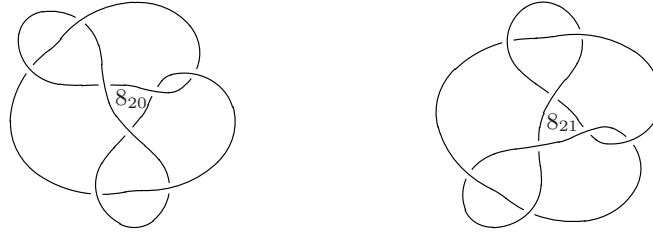
$$(1, 2, 0, -2, -1, 1, 3, 2, 0, -1, 1, 3, 4, 2, 0).$$

Its minimum is -2 and it is achieved only once.

The sequence $f_{r_{20}}(-\chi)$ is the negative of $f_{r_{20}}(\chi)$; so $f_{(r_{20}, -\chi)}$ assumes its minimum once if the maximum of $f_{r_{20}}(\chi)$ is achieved once; this latter condition is fulfilled. All taken together, we see that $\Sigma^1(G) = S(G)$; so the commutator subgroup of G_{20} is finitely generated by Corollary A4.3. Remark B4.5d allows one to say more: G'_{20} is a free group of finite rank.

⁶This data base, created and maintained by C. Livingston, is available at the URL <http://www.indiana.edu/~knotinfo>, January 31, 2013.

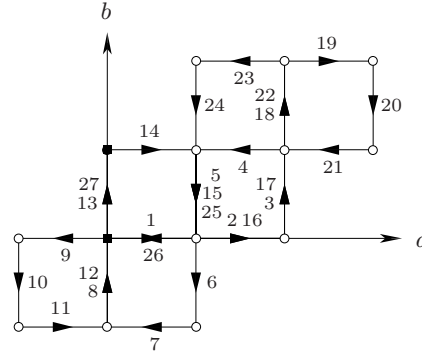
⁷Available at the URL <http://www.geometrygames.org/weeks/>

Figure B.12: Diagrams for knots of types 8_{20} and 8_{21}

Now to the group G_{21} . A run of **SnapPea** has produced a presentation with generators a, b and defining relator

$$r_{21} = a^2 b \cdot a^{-1} b^{-2} \cdot a^{-1} b \cdot a^{-1} b^{-1} \cdot ab^2 \cdot ab^{-1} \cdot ab^2 \cdot ab^{-1} \cdot a^{-1} b \cdot a^{-1} b^{-2} \cdot a^{-1} b. \quad (\text{B4.9})$$

There is no problem in finding a rank 1 character $\chi: G_{21} \rightarrow \mathbb{R}$ and in computing the sequence $f_{r_{21}}(\chi)$; there is, however, an alternative approach which is often more informative and which I want to use in the sequel.

Figure B.13: Path of the relator r_{21} defining the knot group 8_{21}

In this approach, one determines the path \bar{p} in $\Gamma(\mathbb{Z}^2, \{a, b\})$ which corresponds to r_{21} and starts at the origin; this path is displayed in Figure B.13. It has length 27 and ends in $(0, 1)$; so the exponent sum of a in r_{21} is 0 and that of b is 1. The unit vectors giving rise to characters of G_{21} are therefore $u = (1, 0)$ and its opposite. The first vector leads to the linear function $x \mapsto x_1$; along the path, this function is minimal on the 10th edge and maximal on the 20th edge. It follows that $\Sigma^1(G_{21})$ equals $S(G_{21})$.

B4.2a One-relator groups defined by positive words

In [Bau83] G. Baumslag considers the class of one-relator groups whose defining relator r has the form $u \cdot v^{-1}$; here u and v are positive words in the generators

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and the exponent sums of u and v agree for each generator. He proves that such a group is an extension of a free group N by an infinite cyclic group ([Bau83, Thm. 1]). The normal subgroup N need not be finitely generated; indeed it can only be so if G is generated by two elements (see, e. g., Proposition B3.23).

We consider the two-generator case in more detail⁸ but content ourselves with an example, say the group $G = \langle a, b \mid u \cdot v^{-1} \rangle$ with

$$u = a^2 \cdot b \cdot a \cdot b^3 \cdot a^3 \quad \text{and} \quad v = b \cdot a \cdot b \cdot a^4 \cdot b \cdot a \cdot b.$$

The assumption on the exponent sums of u and v imply that the relator $r = u \cdot v^{-1}$ lies in the commutator subgroup of the free group on $\{a, b\}$; so there is an epimorphism $\vartheta: G \rightarrow \mathbb{Z}^2$ which maps (a, b) to the standard basis of \mathbb{R}^2 . The closed path \bar{p} associated to r in $\Gamma(\mathbb{Z}^2, \{a, b\})$ is depicted on the left of Figure B.14.

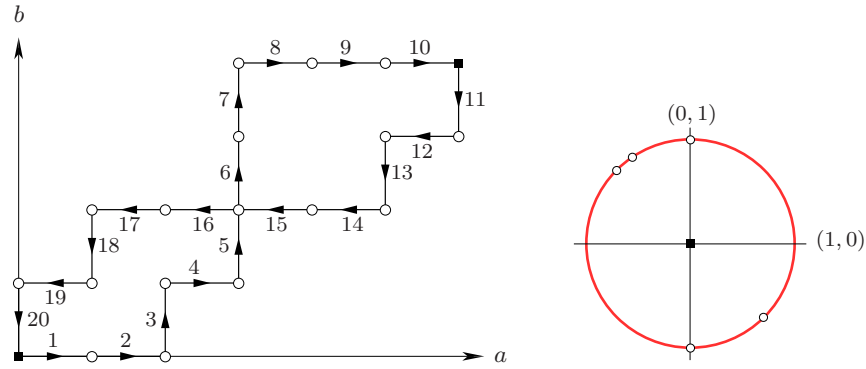


Figure B.14: Path and invariant of a group with positive relation

The words u and v do neither start with nor do they end in the same generator; the relator $r = u \cdot v^{-1}$ is therefore cyclically reduced and the vertices $(0,0)$ and $\vartheta(u) = (6,4)$ are simple. Moreover, the fact that u and v are positive words with the same exponent sums implies that the first and third open quadrants lie in $\Sigma^1(G)$, and so the kernel of every rank 1 character representing a point of the open first quadrant is finitely generated by Corollary A4.3 and free by Remark B4.5d. There is no problem in determining Σ^1 completely; the outcome is depicted on the right of Figure B.14.

⁸The other cases can be reduced to this special case by embedding the group into a two generator group in such a way that the defining relator has still the special form $u' \cdot (v')^{-1}$ where u' and v' are positive words having the same exponent sums for both generators ([Bau83, Lemma 2]).

C Alternate definitions of Sigma 1: Overview

In section A2.1, the invariant Σ^1 has been introduced thus: given a finitely generated group G , choose a finite generating system $\eta: \mathcal{X} \rightarrow G$ and let $\Gamma(G, \mathcal{X})$ denote the associated Cayley graph of G . Consider then a non-zero character $\chi: G \rightarrow \mathbb{R}$ and define G_χ to the submonoid $\{g \in G \mid \chi(g) \geq 0\}$ of G . The Cayley graph $\Gamma(G, \mathcal{X})$ is connected, but its subgraph Γ_χ induced by the monoid G_χ need not be so. Whether or not Γ_χ is connected does not depend on the generating system, it depends only on χ , more precisely on the ray $[\chi]$, and on G . One is thus led to consider the subset $\Sigma^1(G)$ made up of the rays $[\chi]$ with a connected subgraph.

The definition of Σ^1 in terms of Cayley graphs is not the original definition of the invariant $\Sigma_{G'}(G)$ proposed in 1987 by Bieri, Neumann and Strebel. Robert Bieri and Burkhardt Renz, Gaël Meigniez, Ken S. Brown, Gilbert Levitt and Jean-Claude Sikorav found, at about the same time, alternate definitions. Some of these definitions encompass larger, others more restricted, classes of groups. These various definitions show that the concept of Sigma invariant is not only of interest in the area of finitely presented soluble groups, the area which led in 1980 to the introduction of the first such invariant.

In this chapter, and in Chapter D, some of the previously mentioned variations on the definition of Σ^1 will be discussed. In each variation, the Cayley graph is replaced by a structure X on which G acts and satisfies a certain property P . Then the action is restricted to that of the various submonoids G_χ and one collects the points $[\chi] \in S(G)$ for which the action of G_χ continues to enjoy property P .

The variations of this chapter will be grouped into three classes. Those of the first class rephrase in algebraic terms the idea of using G_χ to define a subgraph Γ_χ of a Cayley graph $\Gamma(G, \mathcal{X})$ and asking whether or not Γ_χ is connected. The definitions of the second class take as their starting point the set of path components of the subgraph Γ_χ ; they admit straightforward generalizations to infinitely generated groups. In the variations of the third class the groups under study are fundamental groups G of connected, compact manifolds M and the Cayley graph of G gets replaced by the universal abelian cover \hat{M} of M .

The overview afforded by this chapter is centered on the Cayley graph definition of the invariant: starting from this definition, one moves on to variations of it. This organization must not be construed as implying that the variations arose out of the desire to widen the applicability of the invariant Σ^1 : many of them were invented and investigated independently of the invariant Σ^1 . Only later, did the creators become aware of the relation of their invariants with the invariant $\Sigma_{G'}(G)$ introduced in [BNS87].

C1 Variations for finitely generated groups

A distinctive feature of the variations surveyed in this section is the fact that their natural domain of definition is the class of all finitely generated groups. The first of the variations is, at first sight, hardly more than a simple restatement of the Cayley graph definition; it turns out, however, to be a key opening up new horizons. The second variation is a sufficient condition for a point to lie in Σ^1 ; it allows one, in certain examples, to detect large subsets of Σ^1 with ease. The surprise is that this condition is also necessary.

Notation. Throughout this section, G will denote a finitely generated group, $\mathcal{X} \subset G$ a finite set generating G and $\chi: G \rightarrow \mathbb{R}$ a non-zero character.

C1.1 The invariant $\Sigma^1(G; \mathbb{Z})$

The reformulation of the defining property of Σ^1 , given in this section, will reveal that the invariant $\Sigma^1(G)$ is the first member of an infinite sequence of invariants

$$\Sigma^1(G; \mathbb{Z}) \supseteq \Sigma^2(G; \mathbb{Z}) \supseteq \Sigma^3(G; \mathbb{Z}) \supseteq \cdots.$$

C1.1a The route from $\Sigma^1(G)$ to $\Sigma^1(G; \mathbb{Z})$

We begin with an easy consequence of the defining property of $\Sigma^1(G)$. Suppose the subgraph $\Gamma_{\chi} = \Gamma(G, \mathcal{X})_{\chi}$ is *connected*. Replacing, if need be, some generators $x \in \mathcal{X}$ by their inverses, we may assume that $\chi(\mathcal{X}) \geq 0$. For each element $g \in G_{\chi}$ there exists then a word $w = y_1 y_2 \cdots y_k$ in $\mathcal{Y} = \mathcal{X} \cup \mathcal{X}^{-1}$ which represents g and has the property that the sequence

$$(\chi(y_1), \chi(y_1 y_2), \dots, \chi(y_1 y_2 \cdots y_k)) \quad (\text{C1.1})$$

consists of non-negative real numbers.¹ It follows that the element $g - 1$ of the group ring $\mathbb{Z}G$ admits the expansion

$$\begin{aligned} g - 1 &= y_1 \cdot y_2 \cdots y_k - 1 = (y_1 - 1) + y_1 \cdot (y_2 \cdots y_k - 1) \\ &= (y_1 - 1) + y_1(y_2 - 1) + y_1 y_2 \cdot (y_3 \cdots y_k - 1) = \cdots \\ &= (y_1 - 1) + \sum_{1 \leq j \leq k} y_1 \cdots y_{j-1} (y_j - 1). \end{aligned} \quad (\text{C1.2})$$

In the above, each factor $y_1 \cdots y_j$ lies in the monoid G_{χ} and hence in the monoid ring $\mathbb{Z}G_{\chi}$.²

Let $\varepsilon_{\chi}: \mathbb{Z}G_{\chi} \rightarrow \mathbb{Z}$ denote the augmentation map; it sends each group element $g \in G_{\chi}$ to the number $1 \in \mathbb{Z}$. The kernel $I_{\chi} = \ker \varepsilon_{\chi}$ consists of the linear combinations of differences $g - 1$ with $g \in G_{\chi}$ and it is a two-sided ideal of $\mathbb{Z}G_{\chi}$; in the sequel, we shall view it as a *left* ideal.

The previous calculation shows that the left ideal I_{χ} is generated by the differences $y - 1$ with $y \in \mathcal{X}$ and establishes implication (i) \Rightarrow (ii) in

¹This sequence will be referred to as the χ -*track* of the word $y_1 \cdots y_k$.

²The multiplication of $\mathbb{Z}G_{\chi}$ is inherited from the group ring $\mathbb{Z}G$.

PROPOSITION C1.1 *Let G be a finitely generated group and $\mathcal{X} \subset G$ a finite generating set. For each non-zero character $\chi: G \rightarrow \mathbb{R}$, set $\mathcal{Y}_+ = \{y \in \mathcal{X} \cup \mathcal{X}^{-1} \mid \chi(y) \geq 0\}$. Then the following statements imply each other:*

- (i) $[\chi] \in \Sigma^1(G)$;
- (ii) the left ideal IG_χ is generated by the differences $y - 1$ with $y \in \mathcal{Y}_+$;
- (iii) IG_χ is a finitely generated left ideal.

Proof. Implication (i) \Rightarrow (ii) has been justified in the above and implication (ii) \Rightarrow (iii) clearly holds. So all is well if we can establish implication (iii) \Rightarrow (i).

Suppose $I_\chi = IG_\chi$ is generated by a finite set of elements $\lambda = \sum m_j g_j$. Then $\varepsilon_\chi(\lambda) = \sum m_j$ is 0 and so λ can be rewritten in the $\sum m_j(g_j - 1)$. It follows that there exists a finite set $\mathcal{Z} \subset G_\chi$ such that, for each $g \in G_\chi$, the difference $g - 1$ can be written in the form

$$g - 1 = \sigma_1 g_1(z_1 - 1) + \sigma_2 g_2(z_2 - 1) + \cdots + \sigma_m g_m(z_m - 1) \quad (\text{C1.3})$$

where each σ_j is 1 or -1 , each g_j lies in G_χ and each z_j is an element of \mathcal{Z} .

We shall now prove by induction on the number m of summands that every $g \in G_\chi$ can be represented by a \mathcal{Z}^\pm -word $w = s_1 \cdots s_k$ whose χ -track

$$(\chi(s_1), \chi(s_1 s_2), \dots, \chi(s_1 s_2 \cdots s_k)) \quad (\text{C1.4})$$

is non-negative. Assume first $m = 1$. Then $g - 1 = \sigma_1 g_1(z_1 - 1)$ and two cases arise: if $\sigma_1 = 1$, one has $g_1 = 1$ and $g = g_1 z_1 = z_1$, whence the letter z_1 can be taken as word w ; if $\sigma_1 = -1$ then $g = g_1$ and $g_1 z_1 = 1$, and so $g = z_1^{-1}$. Since g and z_1 both belong to G_χ , the equality $\chi(z_1^{-1}) = 0$ must hold. The word w can therefore be chosen to be z_1^{-1} . Suppose now that $m > 1$. Then g is either a product of the form $g_j z_j$ with $j \in \{1, 2, \dots, m\}$ or one of the g_j . In both cases, there is no harm in assuming that $j = m$. In the first case we obtain the equation

$$g - 1 = \left(\sum_{1 \leq i < m} \sigma_i g_i(z_i - 1) \right) + g_m(z_m - 1) \text{ with } g = g_m z_m,$$

whence $g_m - 1 = \sum_{1 \leq i < m} \sigma_i g_i(z_i - 1)$. Since $g_m \in G_\chi$, the inductive hypothesis applies and thus g_m can be represented by a \mathcal{Z}^\pm -word w' with non-negative χ -track. But if so, $g = g_m z_m$ can be represented by the word $w = w' z_m$ which has non-negative χ -track. Consider, finally, the case where $g = g_m$ and $\varepsilon_m = -1$. Then

$$g - 1 = -g_m(z_m - 1) + \sum_{1 \leq i < m} \sigma_i g_i(z_i - 1)$$

and so $g_m z_m - 1 = \sum_{1 \leq i < m} \sigma_i g_i(z_i - 1)$. It follows, as in the previous case, that there exists a \mathcal{Z}^\pm -word w' with non-negative χ -track that represents $g_m z_m$. The word $w = w' z_m^{-1}$ then represents $g = g_m$ and its χ -track is non-negative, for $\chi(w) = \chi(g) \geq 0$.

The above reasoning implies that \mathcal{Z} generates the group G and shows that the subgraph $\Gamma(G; \mathcal{Z})_\chi$ is connected; so statement (i) is true. \square

REMARKS C1.2 a) The χ -track of the word $w = s_1 \cdots s_k$ (see formula (C1.1)) allows one to describe algebraically whether the associated path $(1, w)$ runs inside the subgraph Γ_χ of the Cayley graph $\Gamma(G; \mathcal{X})$. The definition of the χ -track is identical with that of the sequence $f_r(\chi)$ used in Section B2, but the purpose of that sequence is different: there the word r is a relator of G and one wants to know whether the minimum of the sequence is achieved at most twice, and if assumed twice, whether it is taken at points that are neighbours. In this context, first and last points have to be considered as neighbours.

b) The ideal IG_χ is a two-sided ideal of the monoid ring $\mathbb{Z}G_\chi$. In the above, it is considered it as a left module; if one views IG_χ as a right module over $\mathbb{Z}G_\chi$ its properties as a right module do not correspond to those of the left module $\mathbb{Z}G_\chi$, but to those of the left module $\mathbb{Z}G_{-\chi}$. Indeed, the bijection $g \mapsto g^{-1}$ extends to a linear map $\iota: \mathbb{Z}G \xrightarrow{\sim} \mathbb{Z}G$ which is a ring anti-automorphism. It sends $(g-1) \cdot IG_\chi$ to $\iota(IG_\chi) \cdot (g^{-1}-1) = IG_{-\chi} \cdot (g^{-1}-1)$. A consequence of the stated fact is

LEMMA C1.3 IG_χ is finitely generated qua right $\mathbb{Z}G_\chi$ -module if, and only if, $IG_{-\chi}$ is finitely generated qua left $\mathbb{Z}G_\chi$ -module.

c) Proposition C1.1 (more precisely, a variant of it) is due to R. Bieri and B. Renz (see [BR88, Cor. 6.3]). The proof given in the above seems to be new.

C1.1b The homological invariants $\Sigma^k(G; A)$

Proposition C1.1 shows that the defining property of $\Sigma^1(G)$ can be stated in terms of module theory. This fact becomes more significant if one goes one step further and rephrases the proposition in terms of homological algebra.

The ideal IG_χ is, by definition, the kernel of a ring epimorphism $\varepsilon_\chi: \mathbb{Z}G_\chi \rightarrow \mathbb{Z}$. Now IG_χ can also be viewed as the kernel of the epimorphism of the free cyclic left module $\mathbb{Z}G_\chi$ onto the left module \mathbb{Z} , each element $g \in G_\chi$ acting on \mathbb{Z} by the identity. Schanuel's Lemma³ then shows that the augmentation ideal IG_χ is a finitely generated left ideal if, and only if, the left $\mathbb{Z}G_\chi$ -module \mathbb{Z} admits a finite presentation.

The condition we arrived at is a special case of a well-known homological finiteness condition:

DEFINITION C1.4 Let R be an associative ring with $1 \neq 0$ and A a left R -module. Given $k \geq 0$, one says that A is of type FP_k if there exists an exact sequence

$$\cdots \rightarrow P_{k+1} \rightarrow P_k \rightarrow F_{k-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0 \quad (\text{C1.5})$$

in which each of the modules P_k, P_{k-1}, \dots, P_1 and P_0 is a finitely generated projective R -module.

³see, e. g., [Pas77, p. 431, Lemma 3.1]

A module of type FP_0 is nothing but a finitely generated module; a module is of type FP_1 if, and only if, it admit a finite presentation.

In the terminology just introduced, Proposition C1.1 can be restated as

COROLLARY C1.5 *For every finitely generated group G the following equation is valid:*

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \mathbb{Z} \text{ is of type } \text{FP}_1 \text{ over the ring } \mathbb{Z}G_\chi\}. \quad (\text{C1.6})$$

The condition appearing on the right of equation (C1.6) admits of an obvious generalization to (finitely generated) $\mathbb{Z}G$ -modules A and arbitrary dimensions k . This generalization leads to the invariants $\Sigma^k(G; A)$ introduced by R. Bieri and B. Renz in [BR88]. Here we content ourselves with stating their definition and mentioning one obvious property:

DEFINITION C1.6 Given a finitely generated group G , a finitely generated left $\mathbb{Z}G$ -module and an integer $k \geq 0$, set

$$\Sigma^m(G; A) = \{[\chi] \in S(G) \mid A \text{ is of type } \text{FP}_m \text{ over the ring } \mathbb{Z}G_\chi\}. \quad (\text{C1.7})$$

Then $\Sigma^m(G; A)$ is a subset of the sphere $S(G)$; it is called *homological geometric invariant of G and A in dimension m* .

The new invariants are open subsets of the sphere $S(G)$, as is $\Sigma^1(G)$. This property, however, is far from evident. The only obvious fact about the new invariants is the descending chain of inclusions

$$S(G) \supseteq \Sigma^0(G; A) \supseteq \Sigma^1(G; A) \supseteq \Sigma^2(G; A) \supseteq \cdots \supseteq \Sigma^m(G; A) \supseteq \cdots.$$

REMARKS C1.7 a) Corollary C1.5 asserts that the invariant $\Sigma^1(G)$ coincides with the homological invariant $\Sigma^1(G; \mathbb{Z})$. This fact is already pointed out in [BR88, Proposition 6.4]). The proof given in the above seems to be new.

b) In Section D1, the invariant Σ^0 will be investigated in greater detail.

C1.2 The invariant $\Sigma_{G'}(G)$

In this section, we show that $\Sigma^1(G)$ coincides with the invariant $\Sigma_{G'}(G)$ introduced and studied in [BNS87]. We begin with a restatement of the defining property of Σ^1 that is long overdue.

C1.2a Reexpressing the connectedness of Γ_χ

Suppose the subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected. Then every element of G_χ can be written as an \mathcal{X}^\pm -word with non-negative χ -track. This holds, in particular, for every element g in the commutator subgroup G' . Conversely, assume every element in G' admits a representation as an \mathcal{X}^\pm -word with non-negative χ -track

on the Sigma invariants

and consider an element $g \in G_\chi$. Since \mathcal{X} generates G , there exists a word $w = y_1 y_2 \cdots y_k$ in \mathcal{X}^\pm that represents it. The χ track of this word may be negative, but by rearranging the order of the letters in w we can construct a word $w' = y_{\sigma(1)} \cdots y_{\sigma(k)}$ with non-negative χ -track; it suffices to bring all letters with positive χ -value to the front. The words w and w' represent the same element of G/G' . So there exists a word w'' representing a word of G' with $w \equiv w'w''$. Since w'' can be chosen to have non-negative χ -track this shows that every element $g \in G_\chi$ can be represented by an \mathcal{X} -word with $v_\chi(g) \geq 0$, and so $\Gamma(G, \mathcal{X})_\chi$ is connected.

The preceding argument holds for arbitrary groups, not only for finitely generated ones, a fact that will turn out to be useful in section C2.1a. Since the kernel of a character contains the commutator subgroup of G , we have established

LEMMA C1.8 *Let G be an arbitrary group, $\eta: \mathcal{X} \rightarrow G$ a generating system and $\chi: G \rightarrow \mathbb{R}$ a non-zero character. Then the following statements imply each other:*

- (i) *the subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected;*
- (ii) *for every $g \in \ker \chi$ there exists an \mathcal{X}^\pm -word with non-negative χ -track;*
- (iii) *for every $g \in G'$ there exists an \mathcal{X}^\pm -word with non-negative χ -track.*

C1.2b Relation to $\Sigma^1(G)$

We begin by stating a condition, depending on a finitely generated group G and a non-zero character χ , and then proceed to show that it implies that $[\chi] \in \Sigma^1(G)$.

Given G and χ , consider the condition

$$G' \text{ is generated by a finite subset } \mathcal{A} \text{ over a fg submonoid } M \subset G_\chi. \quad (\text{C1.8})$$

In this statement G' is viewed as a *left* G -operator group, the operation being given by conjugation, thus ${}^g a = g a g^{-1}$.

Assume G and χ satisfy condition (C1.8). Let \mathcal{X} be a finite set generating the finitely generated monoid M and consider an element $g \in G'$. There exist then a sequence of elements g_1, \dots, g_ℓ in M and a sequence of letters a_1, \dots, a_ℓ in $\mathcal{A} \cup \mathcal{A}^{-1}$ so that

$$g = {}^{g_1} a_1 \cdot {}^{g_2} a_2 \cdots {}^{g_\ell} a_\ell$$

By assumption, each element g_j can be expressed by a *positive* word w_j in the alphabet \mathcal{X} . The word

$$w = w_1 a_1 w_1^{-1} \cdot w_2 a_2 w_2^{-1} \cdots w_\ell a_\ell w_\ell^{-1}$$

is then a word in the alphabet $\mathcal{X} \cup \mathcal{X}^{-1} \cup \mathcal{A} \cup \mathcal{A}^{-1}$; it represents g and has non-negative χ -track. In view of Lemma C1.8 the subgraph $\Gamma(G, \mathcal{X} \cup \mathcal{A})_\chi$ is therefore connected and so $[\chi] \in \Sigma^1(G)$.

REMARK C1.9 Condition (C1.8) is the defining property of the invariant $\Sigma_{G'}(G)$, introduced and investigated by R. Bieri, W. D. Neumann and R. Strebel in

Notes

[BNS87], except for the fact that in [BNS87] the group G' is considered as a *right* G -operator group. This implies that the invariant studied there is the image under the antipodal map of the set

$$\Sigma_{G'}(G) = \{[\chi] \in S(G) \mid G' \text{ is fg over a fg submonoid of } G_\chi\}. \quad (\text{C1.9})$$

There is thus a clash of notation between the invariant of [BNS87] and the invariant that will be investigated in this monograph. I prefer to live with this conflict rather than to introduce new notation and to make sure that the reader is never in doubt as to whether left or right action is used in the context at hand.

The insight arrived at so far is summarized by the following

PROPOSITION C1.10 *Given a finitely generated group G , view its commutator subgroup G' as a left G -operator group. Then $\Sigma_{G'}(G) \subseteq \Sigma^1(G)$.*

C1.2c Application: the invariant of Houghton's groups

Proposition C1.10 can be helpful in situations where the action of G on its commutator subgroup G' is sufficiently well known. Instances where this condition is met with are provided by a sequence of permutation groups studied by C. Houghton in [Hou79, pp. 257–258]. These groups generalize example B3.34 b) and are defined like this.

Given a natural number $m \geq 2$, set $\mathcal{S} = \mathcal{S}_m = \mathbb{N} \times \{1, 2, \dots, m\}$; one can think of \mathcal{S}_m as being the disjoint union of m rays that emanate from a point in the plane. Define G_m to be the group of all permutations of \mathcal{S}_m which are eventually a translation. More precisely, a permutation $g: \mathcal{S}_m \xrightarrow{\sim} \mathcal{S}_m$ belongs to G_m if, and only if, there is a vector $x_g = (x_1, x_2, \dots, x_m) \in \mathbb{Z}^m$ such that the equation $g((n, j)) = (n + x_j, j)$ holds for each $j \in \{1, \dots, m\}$ and all sufficiently large $n \in \mathbb{N}$.

The vector x_g is uniquely determined by g and the assignment $g \mapsto x_g$ defines a homomorphism $\tilde{\vartheta}: G_m \rightarrow \mathbb{Z}^m$ whose image coincides with the the subgroup

$$Q_m = \{x \in \mathbb{Z}^m \mid x_1 + x_2 + \dots + x_m = 0\}$$

of \mathbb{Z}^m . Indeed, it is clear that $\text{im } \tilde{\vartheta} \subseteq Q_m$; to prove the opposite inclusion, we construct a collection of “translations” in G_m . For each $j \in \{1, 2, \dots, m\}$, let t_j denote the permutation that fixes the rays $\mathbb{N} \times \{i\}$ with $i \notin \{j, j+1\}$ pointwise and acts on the line formed by the remaining two rays by the rule ⁴

$$t_j(n, j) = \begin{cases} (n-1, j) & \text{if } n \geq 1 \\ (0, j+1) & \text{if } n = 0, \end{cases} \quad \text{and} \quad t_j(n, j+1) = (n+1, j+1). \quad (\text{C1.10})$$

The claim now follows from the facts that $\tilde{\vartheta}(t_j)$ is the vector $e_{j+1} - e_j$ and that these differences generate the subgroup Q_m .

⁴Here and in the sequel, the index j is taken modulo m .

Let $\vartheta: G_m \rightarrow Q_m$ denote the homomorphism obtained from $\tilde{\vartheta}$ by restricting the domain of values to $Q_m = \text{im } \tilde{\vartheta}$. The kernel of ϑ is the group of all finitary permutation of the set $\mathcal{S}_m = \mathbb{N} \times \{1, 2, \dots, m\}$.

PROPOSITION C1.11 *For every $m > 2$, the group G_m is finitely generated. The complement of its invariant $\Sigma^1(G_m)$ consists of m rank 1 points; more precisely*

$$\Sigma^1(G_m)^c = \{[-\chi_1], [-\chi_2], \dots, [-\chi_m]\} \quad (\text{C1.11})$$

where each χ_j is the character sending g to the j -th component $(x_g)_j \in \mathbb{Z}$ of $\vartheta(g)$.

Proof. We first show that $G = G_m$ is generated by the translations t_j defined by equation (C1.10). Let T denote the subgroup generated by them. Then $\vartheta(T) = Q = Q_m$ and so it suffices to verify that T contains the kernel of ϑ . This kernel is the group of all finitary permutations of \mathcal{S}_m , and so generated by transpositions; we claim each transposition is a product of commutators. Indeed, the commutator $[t_{j-1}, t_j] = t_{j-1} \circ t_j \circ t_{j-1}^{-1} \circ t_j^{-1}$ fixes each set $(\mathbb{N} \setminus \{0\}) \times \{i\}$ with $i \in \{1, 2, \dots, m\}$ pointwise, it fixes the endpoints $(0, i)$ with $i \notin \{j, j+1\}$ and, as $m > 2$, it acts on the two remaining endpoints like this

$$\begin{aligned} (0, j) &\mapsto (1, j) \mapsto (0, j) \mapsto (0, j+1) \mapsto (0, j+1), \\ (0, j+1) &\mapsto (0, j) \mapsto (0, j-1) \mapsto (0, j-1) \mapsto (0, j). \end{aligned}$$

So $[t_{j-1}, t_j]$ is the transposition exchanging the endpoints $(0, j)$ and $(0, j+1)$. In view of the action of t_j on the lines $\mathbb{N} \times \{j\} \cup \mathbb{N} \times \{j+1\}$ it follows, firstly, that every transposition of adjacent points (j, n) and $(j, n+1)$ on the j -th ray $\mathbb{N} \times \{j\}$ belongs to T and is a commutator in T , and then that every transposition is a product of commutators in T . We conclude that $\ker \vartheta$ is contained in T' and then that $T = G$.

We move on to the proof of formula (C1.11). The derived group of G_m is generated by the transpositions of adjacent elements of the rays $\mathbb{N} \times \{j\}$ and by the transpositions that exchange the end points of two adjacent rays.

Consider now a non-zero character $\chi: G \rightarrow \mathbb{R}$. Suppose there exists, for each ray $\mathbb{N} \times \{j\}$, an element $g_j \in G_\chi$ such that $\chi(g_j) > 0$; let p_j be a positive integer such that g_j acts on $\{p_j, p_j+1, \dots\} \times \{j\}$ by a translation with amplitude $a_j = \chi(g_j)$. Then G'_m is generated by the transpositions of the finite set

$$\mathcal{F} = \bigcup_{1 \leq j \leq m} \{0, 1, \dots, p_j + a_j\} \times \{j\}$$

and their conjugates under the positive powers of the elements g_1, \dots, g_m . The derived group G'_m is thus finitely generated over the monoid M generated by the elements g_1, \dots, g_m , whence $[\chi] \in \Sigma_{G'_m}(G_m)$ by definition (C1.9) and thus $[\chi] \in \Sigma^1(G_m)$ by Proposition C1.10.

There remains the problem of finding out when there exists a sequence of elements g_1, \dots, g_m with the stated properties. This is a problem in euclidean geometry. Indeed, by introducing coordinates, as detailed in section A1.1d, the

previous problem becomes the following one: let u_1, \dots, u_m be rationally defined, pairwise distinct unit vectors in the standard euclidean space \mathbb{R}^{m-1} . Each u_j gives rise to an open half lattice $\mathcal{H}_j = \{x \in \mathbb{Z}^{m-1} \mid \langle u, x \rangle > 0\}$ of \mathbb{Z}^{m-1} . Then \mathcal{H}_j intersects a closed half space $\mathcal{H}_u = \{y \in \mathbb{R}^{m-1} \mid \langle u, y \rangle \geq 0\}$ non-trivially if, and only if, $u \neq -u_j$. The previous argument therefore implies that

$$\Sigma^1(G_m)^c \subseteq \{[-\chi_1], [-\chi_2], \dots, [-\chi_m]\}.$$

We are left with proving that the points $-\chi_1, \dots, -\chi_m$ lie in $\Sigma^1(G_m)^c$. Given an index $j \in \{1, 2, \dots, m\}$, consider the subgroup H_j of G_m made up of all permutations that fix the ray $\mathbb{N} \times \{j+1\}$ pointwise. The translation t_j maps the set $S_j = \mathcal{S}_m \setminus \mathbb{N} \times \{j+1\}$ onto $S_j \cup \{(0, j+1)\}$ and the images of S_j under the positive powers of t_j sweep out all of \mathcal{S}_m . It follows that G_m is a strictly ascending HNN-extension with base group H_j and stable letter t_j , whence G_m is a strictly *descending* HNN-extension with base group H_j and stable letter t_j^{-1} . Since $(-\chi_j)(t_j^{-1}) = 1$, Proposition B2.9 allows us to see that $-\chi_j \notin \Sigma^1(G_m)$. \square

NOTE C1.12 The invariant of Houghton group G_m has been worked out by Ken Brown around 1985 by means of his characterization of Σ^1 in terms of actions on \mathbb{R} -trees (see [Bro87b, Section 5]). The proof given here was found by R. Bieri and R. Strebel at about the same time.

C1.2d Equality of $\Sigma^1(G)$ and $\Sigma_{G'}(G)$

Proposition C1.10 states that the invariant $\Sigma_{G'}(G)$, propounded in [BNS87], is a subset of $\Sigma^1(G)$. Actually the reverse inclusion also holds:

THEOREM C1.13 *For every finitely generated group G the invariant $\Sigma^1(G)$ coincides with*

$$\Sigma_{G'}(G) = \{[\chi] \in S(G) \mid G' \text{ is fg over a fg submonoid of } G_\chi\}. \quad (\text{C1.12})$$

Proof. The inclusion $\Sigma_{G'}(G) \subseteq \Sigma^1(G)$ is covered by Proposition C1.10; the reverse inclusion will be established in two steps, summarized by the formula

$$[\chi] \in \Sigma^1(G) \implies \text{gp}(\mathcal{W}(\mathcal{X}, \chi) \mathcal{A}) = G' \implies [\chi] \in \Sigma_{G'}(G). \quad (\text{C1.13})$$

In the above, the following notation is used: $\mathcal{X} \subset G$ is a finite set of generators of G , next \mathcal{Y} is the alphabet $\mathcal{X} \cup \mathcal{X}^{-1}$ and $\mathcal{W}(\mathcal{X}, \chi)$ denotes the set of all \mathcal{Y} -words with non-negative χ -track. Finally, \mathcal{A} denotes the set of all commutators

$$[y_1, y_2] = y_1 y_2 y_1^{-1} y_2^{-1} \text{ with } y_1 \text{ and } y_2 \text{ in } \mathcal{Y}.$$

Let \mathcal{X} , \mathcal{Y} and \mathcal{A} are as explained before and let $\chi: G \rightarrow \mathbb{R}$ a non-zero character that represents a point of $\Sigma^1(G)$. Every element $g \in G'$ can then be represented

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by a \mathcal{Y} -word w' with non-negative χ -track (cf. implication (i) \Rightarrow (iii) of Lemma C1.8). We intend to rewrite w' as a product w of conjugates of the commutators in \mathcal{A} . If such a product exists, it will have zero exponent sum with respect to every generator $x \in \mathcal{X}$; the word w' , however, may not have this property. One can remedy such a default as follows.

Let η denote the obvious projection of F onto G . The preimage $U = \eta^{-1}(G')$ contains the derived group F' of F ; let \mathcal{U}' be a finite set of words in \mathcal{Y} that generates U modulo F' . The image $\eta(u')$ of each $u' \in \mathcal{U}'$ lies in G' ; since η maps F onto all of G , there exists a word $w_{u'} \in F'$ with $\eta(u') = \eta(w_{u'})$. By construction, each word in the finite set $\mathcal{U}' \cup \{w_{u'} \mid u' \in \mathcal{U}'\}$ represents an element of G' , but its χ -track may be negative. There exists, however, a letter $y \in \mathcal{Y}$ with $\chi(y) > 0$ and an exponent $k > 0$ so that the χ -tracks of all the elements in

$$\mathcal{U} = \{u = y^k u' y^{-k} \mid u' \in \mathcal{U}'\} \quad \text{and} \quad \{w_u = y^k w_{u'} y^{-k} \mid u' \in \mathcal{U}'\}$$

are 0. Let now g be an element of G' . As $[\chi] \in \Sigma^1(G)$ there exists a word $w' \in F$ with $\eta(w') = g$ and χ -track equal to 0. Next, there exist a product $u_1 \cdots u_k$ of words in \mathcal{U} so that $w'' = w' \cdot u_1 \cdots u_k$ has exponent sum 0 for every $x \in \mathcal{X}$. The χ -track of the word w'' is then 0, as is the χ -track of the word $w = w'' \cdot (w_{u_1} \cdots w_{u_k})^{-1}$. Notice that the word w represents the given element $g \in G'$ and that it lies in F' , as we set out to show.

We are now ready to justify the first implication in (C1.13). For every $g \in G'$ the previous argument provides one with a \mathcal{Y} -word $w \in F'$ and χ -track equal to 0. We intend to show by induction on the number m of letters in w that w is freely equivalent to a word of the form

$$v_1 a_1 \cdot v_2 a_2 \cdots v_f a_f$$

where each v_j is a \mathcal{Y} -word with non-negative χ -track and each a_j is in \mathcal{A} . The claim is obvious if $m = 0$. If $m > 0$, consider the leftmost letter z in w with $\chi(z) \leq 0$. This letter is either preceded or succeeded by an occurrence of z^{-1} , for the exponent sums of w are equal to 0. In the first case, set $y = z^{-1}$; in the second case, set $y = z$. Then w has the form $w_1 y w_2 y^{-1} w_3$ and so it is freely equivalent to

$$w_1 \cdot y w_2 y^{-1} w_2^{-1} \cdot w_2 w_3 \equiv {}^{w_1}[y, w_2] \cdot w_1 w_2 w_3.$$

The word $w_1 w_2 w_3$ is shorter than w and its exponent sums are 0. Its χ -track is likewise 0: if $\chi(y) \leq 0$, this assertion is obvious; if not, each letter in $w_1 w_2$ has a positive χ -value by the choice of z , and so the assertion holds likewise. By the inductive assumption, the word $w_1 w_2 w_3$ represents therefore an element in the subgroup $\text{gp}^{(\mathcal{X}, \chi)}(\mathcal{A})$. Let $y_1 \cdots y_k$ be the spelling of w_2 . The commutator identity $[a, bc] = [a, b] \cdot {}^b[a, c]$ implies then that ${}^{w_1}[y, w_2]$ is freely equivalent to

$${}^{w_1}([y, y_1] \cdot {}^{y_1}[y, y_2] \cdots {}^{y_1 \cdots y_{k-1}}[y, y_k]) \equiv {}^{w_1}[y, y_1] \cdot {}^{w_1 y_1}[y, y_2] \cdots {}^{w_1 y_1 \cdots y_{k-1}}[y, y_k].$$

Let w_0 denote the word that stands on the right hand side of this equivalence. As each of the conjugating words $w y_1 \cdots y_j$ occurring w_0 is an initial word of $w_1 w_2$ it

has non-negative χ -track. Hence w_0 represents a word in the group $\text{gp}(\mathcal{W}^{(\mathcal{X}, \chi)} \mathcal{A})$. The inductive step of the first implication in (C1.13) is now complete.

In the second part, the representation obtained in the first step, is applied to the commutators ${}^y a$ with $y \in \mathcal{Y} = \mathcal{X}^\pm$ and $a \in \mathcal{A} = \{[y_1, y_2] \mid (y_1, y_2) \in \mathcal{Y}^2\}$. Let

$${}^y a = {}^{w_1} a_1 \cdot {}^{w_2} a_2 \cdots {}^{w_f} a_f \quad (\text{C1.14})$$

be the expression resulting for ${}^y a$ and let $\mathcal{I}(y, a)$ be the set of all initial words of the conjugating words w_j occurring in this expression. Define \mathcal{I} to be the union of these finite sets $\mathcal{I}(y, a)$ and let M be the monoid generated by the image of \mathcal{I} in the group G . We claim that \mathcal{A} generates G' over M .

Let $H \subseteq G'$ denote the subgroup generated by the set ${}^M \mathcal{A}$. Since G' is generated by \mathcal{A} over G , the equality of G' and H will follow if H is stable under conjugation by the elements of \mathcal{Y} . Let $w = y_1 \cdots y_k$ be a \mathcal{Y} -word that is a product of words in \mathcal{I} . By the construction of \mathcal{I} , each initial segment w' of w is a product of words in \mathcal{I} . Given $a \in \mathcal{A}$ and $y \in \mathcal{Y}$, we now rewrite ${}^{yw} a$ as follows:

$$\begin{aligned} {}^{yy_1 \cdots y_k} a &= {}^{yy_1} y^{-1} y_1^{-1} \cdot {}^{y_1 y y_2 y_k} a \\ &= [y, y_1] \cdot {}^{y_1 y y_2 \cdots y_k} a \cdot [y, y_1]^{-1} = a_1 \cdot {}^{y_1 y y_2 \cdots y_k} a \cdot a_1^{-1}. \end{aligned}$$

Continuing in this manner we end up with the equation

$${}^{yy_1 \cdots y_k} a = (a_1 \cdot {}^{y_1} a_2 \cdots {}^{y_1 \cdots y_{k-1}} a_k) \cdot ({}^{y_1 y_2 \cdots y_k} a) \cdot (a_1 \cdot {}^{y_1} a_2 \cdots {}^{y_1 \cdots y_{k-1}} a_k)^{-1}.$$

The first and the third factor of the right hand side are contained in H since the occurring conjugating words are initial segments of w ; the second factor lies in H by relation (C1.14) and the definition of M . So ${}^y H \subseteq H$, as desired. \square

NOTE C1.14 The justification for the second implication in formula (C1.13) goes back to the proof of implication (iv) \Rightarrow (i) in [BNS87, Proposition 2.1].

We conclude section C1.2 with two applications of the theorem just proved.

C1.2e A characterization of the complement of Σ^1

So far, various results have been established that allow one show to that a given character $\chi: G \rightarrow \mathbb{R}$ represents a point of the invariant Σ^1 . By contrast, only three general results are at one's disposal if one sets out to prove that a character $\chi: G \rightarrow \mathbb{R}$ does *not* represent a point of Σ^1 . The first of them deals with the invariant of a non-abelian free group or, more generally, of a non-trivial free product $G = G_1 \star G_2$ and states that the invariant is empty (see example 3) in section A2.1a). This result will be generalized in section C2.1d; there it will be shown that if G is a free product with amalgamation $G_1 \star_A G_2$ with A distinct from both G_1 and G_2 then $S(G, A) \subseteq \Sigma^1(G)^c$ (see part (i) of Proposition C2.13).

Secondly, one knows that in case G maps onto a quotient group Q every point $[\psi] \in \Sigma^1(Q)^c$ pulls back to a point in $\Sigma^1(G)^c$ (see Corollary B1.8). This observation is used, for instance, in the proofs of Theorem B1.17 or of Proposition B3.23.

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Thirdly, Proposition B2.9 states that the canonical character of a *strictly descending HNN-extension* represents a point in the complement of Σ^1 ; this result deals only with rank 1 characters. Theorem C1.13 allows one to generalize it as follows:

PROPOSITION C1.15 *Let G be a finitely generated group and let χ be a non-zero character of G . Consider a subgroup N of G that contains G' and is contained in $\ker \chi$. Then $[\chi] \in \Sigma^1(G)^c$ if, and only if, N contains an increasing chain $B_0 \subseteq B_1 \subseteq \cdots$ of proper subgroups with the following two properties:*

$$(i) \bigcup_{j \in \mathbb{N}} B_j = N;$$

$$(ii) \text{ for every } g \in G_\chi \text{ there exists an index } j(g) \text{ such that the inclusion } g \cdot B_j \cdot g^{-1} \subseteq B_j \text{ holds for each } j \geq j(g).$$

Proof. Assume first a chain of subgroups B_j satisfying requirements (i) and (ii) exists. Then every subgroup B of G' that is finitely generated over a finitely generated submonoid $M \subseteq G_\chi$ is contained in B_j for all sufficiently large indices j . Since no member B_j of the chain $B_0 \subseteq B_1 \subseteq \cdots$ can contain G' — for each member is a proper subgroup and N/G' is finitely generated — Theorem C1.13 allows one to conclude that $[\chi] \notin \Sigma^1(G)$.

Conversely, assume that $[\chi] \notin \Sigma^1(G)$. Let $\eta: \mathcal{X} \rightarrow G$ be a finite generating system of G and $\Gamma = \Gamma(G, \mathcal{X})$ the associated Cayley graph of G . Define B_j to be the set of elements $g \in N$ that can be represented by a word with χ -track bounded from below by $-j$. Lemmata A2.9 and C1.8, imply that each of these subsets is distinct from N . The sets B_j are subgroups of N (cf. Remark A2.2) and they form an ascending chain whose union is N . Consider now an element $g \in G_\chi$. Represent it by some word $w(g)$ with χ -track bounded from below by some integer, say $-j(g)$. Then $gB_jg^{-1} \subseteq B_j$ for every index $j \geq j(g)$. The subgroups B_j are therefore proper subgroups of N and they form an increasing chain that satisfies requirements (i) and (ii) stated in Proposition C1.15. \square

NOTE C1.16 Proposition C1.15 is a variant of Proposition 9.1 of [BNS87].

C1.2f Application: Σ^1 of a wreath product

In section A2.3b, the invariant of a direct product $H \times Q$ of two finitely generated groups has been determined. The outcome is summarized in Proposition A2.7; it shows that $\Sigma^1(H \times Q)^c$ is, in essence, the union of $\Sigma^1(H)^c$ and $\Sigma^1(Q)^c$. Below, we shall establish a formula for the invariant of the wreath product $G = H \wr Q$ of two finitely generated groups H and Q . This formula differs from that for a direct product in an important aspect: $\Sigma^1(G)^c$ depends only on the abelianisations of H and of Q .

DEFINITION C1.17 The (restricted standard) wreath product of the group H by the group Q is the quotient of the free product $H \star Q$ modulo the normal subgroup generated by the commutators

$$[h_1, qh_2q^{-1}] = h_1 \cdot qh_2q^{-1} \cdot h_1^{-1} \cdot qh_2^{-1}q^{-1}; \quad (\text{C1.15})$$

here (h_1, h_2) ranges over H^2 and q over $Q \setminus \{1\}$. The wreath product of H by Q will be denoted by $H \wr Q$.

Definition C1.17 describes $H \wr Q$ by a presentation. Alternatively, one can define the wreath product by an explicit construction: it is the semi-direct product $L \rtimes Q$ of the so called base group $L = \text{gp}_G(H)$ by Q where L denotes the restricted direct product $\text{Dr}\{qHq^{-1} \mid q \in Q\}$ of $\text{card}(Q)$ copies of H .

In the sequel, H and Q will be assumed to be finitely generated. If one of them is reduced to the unit element, the wreath product is nothing but the other group; this case presents no interest and will be excluded in the sequel.

The definition of $H \wr Q$ implies that there is an epimorphism $\pi: H \wr Q \twoheadrightarrow Q$ which sends $h \in H$ to $1 \in Q$ and Q onto itself by the identity. This epimorphism gives rise to the embedding $\pi^*: S(Q) \hookrightarrow S(H \wr Q)$. In addition, there is an epimorphism $\bar{\rho}$ of $H \wr Q$ onto H_{ab} ; it maps $h \in H$ to hH' and $q \in Q$ to $1H'$. The epimorphisms $\bar{\rho}$ and π give rise to an isomorphism groups $(H \wr Q)_{\text{ab}} \xrightarrow{\sim} H_{\text{ab}} \times Q_{\text{ab}}$ and to an isomorphism of spheres

$$S(H_{\text{ab}} \times Q) \xrightarrow{\sim} S(H \wr Q).$$

It follows, in particular, that $S(H \wr Q)$ is empty if the abelianisations of H and Q are both finite; so there exist wreath products that are of little interest in the context of Σ^1 , but which are admitted in Proposition C1.18 below.

This proposition describes the *complement* of the invariant of the wreath product of two arbitrary finitely generated groups, the sole restriction being that neither of the factors H and Q be the trivial group.

PROPOSITION C1.18 *Let G be the wreath product $H \wr Q$ of two finitely generated, non-trivial groups H and Q . Then $\Sigma^1(G)^c = S(G, H)$.*

Proof. Let $\chi: G \rightarrow \mathbb{R}$ be a non-zero character. The proof splits into three cases, depending on as to whether χ vanishes on H , on Q , or on neither of them.

Assume first that χ *vanishes on H* . The aim is to construct a sequence of subgroups $B_0 \subset B_1 \subset \dots$ in the kernel N of χ and to deduce, with the help of Proposition C1.15, that $[\chi] \notin \Sigma^1(G)$. Put $Q_0 = Q \cap \ker \chi$ and

$$B_j = \text{Dr}\{qHq^{-1} \mid \chi(q) \geq -j\} \rtimes Q_0.$$

Then each of these sets B_j is a proper subgroup of $N = \ker \chi$ and their union is N . Moreover, each B_j is invariant under conjugation by the submonoid $Q_{\chi|Q}$. As

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each element $g \in G_\chi$ is a product of the form $n \cdot q$ with $n \in N$ and $q \in Q_{\chi|_Q}$, Proposition C1.15 therefore applies and shows that $[\chi] \notin \Sigma^1(G)$.

Suppose next that χ *vanishes on* Q . The plan is to show that the normal subgroup $N = G' \cdot Q$ is finitely generated. The Addendum to Corollary A4.5 will then imply that $S(G, Q) \subseteq \Sigma^1(G)$. Let B denote the base group of G , i. e., the direct sum $\text{Dr}\{qHq^{-1} \mid q \in Q\}$. Choose a finite set of generators \mathcal{H} of H , and a finite set of generators \mathcal{Q} of Q ; we can and shall assume that $\mathcal{Q} \subset Q \setminus \{1\}$. Set

$$\mathcal{C} = \{[h_1, h_2] \mid h_i \in \mathcal{H}\}, \quad (\text{C1.16})$$

$$\mathcal{K} = \{[h, q] \mid (h, q) \in \mathcal{H}^\pm \times \mathcal{Q}\}, \text{ and} \quad (\text{C1.17})$$

$$L = \text{gp}(\mathcal{C} \cup \mathcal{K} \cup \mathcal{Q}). \quad (\text{C1.18})$$

Then L is a finitely generated subgroup of N ; we are going to prove that L equals N whence $N = G' \cdot Q$ is finitely generated.

We first show that L contains the derived group $B' = \text{Dr}\{qH'q^{-1} \mid q \in Q\}$ of the base group B . The group H' is the normal closure in H of the subset $\mathcal{C} \subset L$. Since conjugation by $[h, q] = h \cdot qh^{-1}q^{-1}$ has the same effect on H as conjugation by h whenever $q \in Q \setminus \{1\}$ and as $\mathcal{K} \cup \mathcal{Q} \subset L$, it follows, first, that $H' \subset L$ and then that $B' \subset L$.

Now B' is a normal subgroup of G and G/B' is isomorphic to $B_{\text{ab}} \rtimes Q$; it suffices therefore to prove that $\bar{L} = L/B'$ coincides with $\bar{N} = N/B'$. The derived group of $\bar{G} = G/B'$ has the form $M \rtimes Q'$ where $M \subset B_{\text{ab}}$ is the submodule generated the elements $(1 - q) \cdot \bar{b}$ with $q \in Q$ and $\bar{b} \in B_{\text{ab}}$. Since $N = G' \cdot Q$, this shows that $\bar{N} = M \rtimes Q$. But $B_{\text{ab}} = \mathbb{Z}Q \otimes H_{\text{ab}}$ and the augmentation ideal $IQ = \ker(\mathbb{Z}Q \rightarrow \mathbb{Z})$ is generated by the finite set $\{1 - q \mid q \in Q\}$. The claim thus follows from the fact that $((1 - q) \otimes h) \cdot B'$ is the additive notation of the element $[h, q] \cdot B' \in \mathcal{K} \cdot B'$ and so it lies in L/B' by the definition of L .

Consider, finally, the case where $\chi|_Q$ and $\chi|_H$ are both non-zero. Choose a finite set of generators \mathcal{H} of H and a finite set of generators \mathcal{Q} of Q so that χ is positive on each of the elements of $\mathcal{X} = \mathcal{H} \cup \mathcal{Q}$. The set \mathcal{X} generates G ; we use it and Proposition B2.15 to show that $[\chi] \in \Sigma^1(G)$.

Pick elements $h \in \mathcal{H}$ and $q \in \mathcal{Q}$. Then $q \neq 1$ and so the words

$$r_{h,q} = [h, qhq^{-1}] = h \cdot qhq^{-1} \cdot h^{-1} \cdot qh^{-1}q^{-1}$$

are relators of G . The minimum of the χ -track of each of these words is 0 and it occurs once, at the end of the word. In view of definition B2.13 and Proposition B2.15, these facts imply that $[\chi] \in \Sigma^1(G)$. In more detail:

Put $\mathcal{R} = \{[h, qhq^{-1}] \mid h \in \mathcal{H}, q \in \mathcal{Q}\}$. Then the set $\mathcal{R}_{\chi,+}$, occurring in definition B2.13, coincides with \mathcal{R} and the graph $\mathcal{G}_{\mathcal{R},\chi}$ is the bipartite graph with \mathcal{H} and \mathcal{Q} as sets of vertices. This graph is connected, for $\chi|_H \neq 0$ and $\chi|_Q \neq 0$, and so the sets \mathcal{H} and \mathcal{Q} are non-empty. By definition B2.13, the point $[\chi]$ thus belongs to $\Psi(\mathcal{R})$ and hence to $\Sigma^1(G)$ by Proposition B2.15. \square

REMARKS C1.19 a) The wreath product allows one to construct groups that are fairly easy to analyze, but have interesting properties. One such construction goes back to Philip Hall's paper [Hal54]. In section 2.4, Hall proves that the wreath product $G = H \wr Q$ of two finitely generated groups satisfies the *maximal condition max-n on normal subgroups* if the first factor H has this property and if the group ring $\mathbb{Z}Q$ of the second factor Q is right (or left) noetherian. As an infinite cyclic group C is noetherian and its group ring is likewise noetherian, this result permits one to construct a chain of finitely generated subgroups

$$G_1 = C, \quad G_2 = C \wr C, \quad G_3 = G_2 \wr C = (C \wr C) \wr C, \dots, \quad G_m = G_{m-1} \wr C, \dots$$

which all satisfy max-n. Let x_1, x_2, \dots, x_m be elements in G_m that generate the infinite cyclic groups involved in the successive wreath products making up G_m . The abelianisation of G_m is free abelian of rank m , generated by the canonical images of the x_j , and, for $m > 1$, the complement of $\Sigma^1(G_m)$ is the 0-dimensional sphere $S(G_m, \text{gp}(x_1, \dots, x_{m-1}))$. The derived length of G_m is m , as one can see by the following inductive argument. The group G_1 is a non-trivial abelian group and so of derived length 1. Assume, inductively, that G_{m-1} has derived length $m-1$. Since G_m is an extension of the direct product $B = \text{Dr}_j \{x_m^j G_{m-1} x_m^{-j}\}$ by the infinite cyclic group $\text{gp}(x_m)$, the derived length of G_m is at most m . On the other hand, the derived group of G contains the commutators $[x_1, x_m], \dots, [x_{m-1}, x_m]$. They generate a subgroup H of $G_{m-1} \oplus x_m G_{m-1} x_m^{-1}$ which projects onto the first factor G_{m-1} . Hence the derived length of G_m is at least m .

b) Suppose H is a non-trivial abelian group. The base group $A = \text{Dr}_j \{qHq^{-1}\}$ of $G = H \wr Q$ is then a non-trivial abelian normal subgroup. Proposition C1.18 shows that every character $\chi: G \rightarrow \mathbb{R}$ with $\chi(A) \neq \{0\}$ represents a point of $\Sigma^1(G)$. This conclusion holds in greater generality:

LEMMA C1.20 *Let G be a finitely generated group with an abelian normal subgroup $A \triangleleft G$. Then $S(G, A)^c$ is contained in $\Sigma^1(G)$.*

Proof. We use a variation of the argument employed in the third part of the proof of Proposition C1.18. Suppose $\chi: G \rightarrow \mathbb{R}$ does not vanish on A and $t \in A$ is an element with $\chi(t) > 0$. There exists then a finite generating system \mathcal{X} that includes t and satisfies $\chi(x) > 0$ for every $x \in \mathcal{X}$. For each $x \in \mathcal{X}$ the commutator $r_x = t \cdot xtx^{-1} \cdot t^{-1} \cdot tx^{-1}t^{-1}$ is then a relator of G whose χ -track has positive values on every proper initial segment, whence $[\chi] \in \Sigma^1(G)$ by Theorem B2.1. \square

Lemma C1.20 will be generalized in section C2.6b; see Corollary C2.53.

c) Random walks on wreath products exhibit unusual behaviour; see, e. g., [PSC02] and the references cited therein. In the literature on random walks some of these wreath products, in particular the metabelian group $(\mathbb{Z}/2\mathbb{Z}) \wr C_\infty$, are called *lamplighter groups*.

C2 Extensions to infinitely generated groups

The variations treated in this section have two features in common:

- a) their definition makes sense for arbitrary groups G , and
- b) they allow one to establish properties of Σ^1 with ease that are awkward to prove in the set-up of Chapter A.

It turns out that the Cayley graph definition of Σ^1 admits also of a straightforward extension to infinitely generated groups. We start out with this extension and then discuss some of its properties and applications. In section C2.2 we move on to the variation put forward and investigated by Gaël Meigniez in [Mei87], [Mei88] and [Mei90]. Then we discuss a characterization due to Ken Brown (see [Bro87b]) (sections C2.3 through C2.5). In section C2.6, finally, we list some consequences of this characterization.

C2.1 Extending Σ^1 to arbitrary groups

We begin by explaining how the definition of Σ^1 in terms of Cayley graph can be adapted to infinitely generated groups. This extension will then be shown to coincide with the alternative definition of Gaël Meigniez and also with that of Ken Brown. The Cayley graph definition will allow one to comprehend why the definitions of Meigniez and Brown are often easier to work with than the original definition, a fact that is true even for finitely generated groups.

C2.1a The generalization

Let G denote an arbitrary group, $\eta: \mathcal{X} \rightarrow G$ a generating system and $\chi: G \rightarrow \mathbb{R}$ a non-zero character of G . As in sections A1.2b and A2.1, one can then define the Cayley graph $\Gamma(G, \mathcal{X})$ and the full subgraph $\Gamma(G, \mathcal{X})_\chi$ induced by the submonoid $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$. The Cayley graph is connected, while its subgraph $\Gamma(G, \mathcal{X})_\chi$ need not be so. Whether it is connected depends on χ — this is intended — but also on η . To get rid of this undesired dependence, we shall work, not with a single generating system, but with all of them.

Whether this manner of getting around a difficulty allows one to derive useful results remains to be seen. Here we are first concerned with the question whether the proposed definition is compatible with the original one. The affirmative answer is given by

LEMMA C2.1 *For every finitely generated group G and every non-zero character $\chi: G \rightarrow \mathbb{R}$ the following statements are equivalent:*

- (i) *there exists a finite generating system $\eta: \mathcal{X}_f \rightarrow G$ for which the subgraph $\Gamma(G, \mathcal{X}_f)_\chi$ is connected,*
- (ii) *the subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected for every generating system $\eta: \mathcal{X} \rightarrow G$.*

Notes

Proof. Assume $\eta: \mathcal{X}_f \rightarrow G$ is a finite generating system for which $\Gamma(G, \mathcal{X}_f)_\chi$ is connected, and let \mathcal{X} is a generating system of G . Since G is finitely generated \mathcal{X} contains a finite subset \mathcal{X}_0 which generates G . Theorem A2.3 then shows that the graph $\Gamma(G, \mathcal{X}_0)_\chi$ is connected. Since $\Gamma(G, \mathcal{X}_0)_\chi$ is a subgraph of $\Gamma(G, \mathcal{X})_\chi$ and as both graphs have the same set of vertices, the graph $\Gamma(G, \mathcal{X})_\chi$ is connected, too.

The previous argument shows that (i) implies (ii). The converse is obvious. \square

In view of Lemma C2.1, the following definition of the invariant Σ^1 for arbitrary groups extends the definition given in section A2.1:

DEFINITION C2.2 Given a group G , let $S(G)$ denote the set of rays $[\chi]$ in the vector space $\text{Hom}(G, \mathbb{R})$ that emanate from the origin and put

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \Gamma(G, \mathcal{X})_\chi \text{ is connected for every gen. system } \mathcal{X}\}. \quad (\text{C2.1})$$

REMARKS C2.3 a) In section A2.1, the invariant Σ^1 has only been defined for *finitely generated groups* G . This restriction is responsible for two hallmarks of the theory treated up to now:

- the vector space $\text{Hom}(G, \mathbb{R})$ is finite dimensional and thus carries a canonical topology;
- there exists a finitary condition \mathcal{FC} that characterizes the points $[\chi] \in \Sigma^1(G)$.

This finitary condition \mathcal{FC} — introduced in Section A3 and called Σ^1 -*criterion* — implies that the invariant is an open subset of the sphere $S(G)$; in addition, it is the main tool in the proof of many basic results, but also of various explicit calculations, of Σ^1 treated in Chapters A and B: see, e. g., Theorems A4.1, B2.1, B4.1, Propositions A3.4, B2.5 and B2.15 for results, and section B2.3b for examples.

b) The preceding chapters contain, however, also results which are obtained by direct geometric arguments. Instances of such results are sprinkled over the first two chapters, starting with the very first examples (abelian groups and non-trivial free products in section A2.1a), the computation of the invariant of a direct product in example B1.9, that of a subgroup of finite index (Proposition B1.11) or that of a join of subgroups (Proposition B1.15). In these instances the finite generation of the groups plays a minor rôle; so the question arises whether this assumption can dispensed with.

c) There are other reasons which prompt one to generalize the invariant to infinitely generated groups G , in spite of the fact that for such a group the vector space $\text{Hom}(G, \mathbb{R})$ can be infinite dimensional and, if so, does not carry a canonical topology. One reason is that a generalization to infinitely generated groups puts one in a position to obtain results for a finitely generated group by studying certain of its infinitely generated subgroups. In addition, the constructions treated in sections C2.2 through C2.4 produce invariants that coincide with Σ^1 for finitely generated groups ⁵ but have as their natural habitat the class of all groups.

⁵See [Mei90, Théorème 3.19] and [Bro87b, Theorem 5.2]

To help the reader with getting used to determine the invariant of an arbitrary group, we continue with the determination of Σ^1 for four classes of examples.

C2.1b Illustration 1: groups with no free submonoids of rank 2

Let G be a group and $\chi: G \rightarrow \mathbb{R}$ a non-zero character. In order to prove that $[\chi]$ lies in $\Sigma^1(G)$, one has to verify that the subgraphs Γ_χ of an infinite number of Cayley graphs $\Gamma(G, \mathcal{X})$ are connected. In this section and the following one, we show that this verification is very easy if G is a group that does not contain a free submonoid of rank 2 or if G is an Engel group.

The basis of our verification will be the following

LEMMA C2.4 *Let χ be a non-zero character of the group G . Assume that for every couple (x_1, x_2) of elements in G with $\chi(x_1) = \chi(x_2) > 0$ there exists words u_1 and u_2 in $\{x_1, x_1^{-1}, x_2, x_2^{-1}\}$ such that*

$$\begin{aligned} x_1 \cdot u_1 \text{ and } x_2 \cdot u_2 \text{ represent the same element in } G \text{ and} \\ \text{the } \chi\text{-tracks of the words } u_1 \text{ and } u_2 \text{ are non-negative.} \end{aligned} \quad (\text{C2.2})$$

Then $[\chi] \in \Sigma^1(G)$.

Proof. Let $\eta: \mathcal{X} \rightarrow \mathbb{R}$ be a generating system of G and Γ the corresponding Cayley graph of G . For every element $g \in \ker \chi \setminus \{1_G\}$ there exists a path $p = (1_G, w)$ in Γ that leads from 1_G to g ; we may assume that p is simple. Let h be a vertex of p with minimal χ -value and let w_1 the initial segment of w that describes the part of p from 1_G to h ; similarly, let w_2 the terminal segment of w that describes the part of p from h to g . If $\chi(h) = 0$, then p runs inside the subgraph Γ_χ and shows that g belongs to the component \mathcal{C}_1 of 1 in Γ_χ .

Otherwise, set $x_1 = h^{-1}$ and $x_2 = h^{-1} \cdot g$. Then $\chi(x_1) = \chi(x_2) > 0$; by hypothesis there exists therefore words u_1, u_2 which satisfy requirement (C2.2). Let \tilde{u}_1, \tilde{u}_2 be the words in $\mathcal{X} \cup \mathcal{X}^{-1}$ that are obtained from u_1 and u_2 , respectively, by replacing each occurrence of the letter x_1^ε by the word $w_1^{-\varepsilon}$ and each occurrence of the letter x_2^ε by w_2^ε . Since h has minimal χ -value among the vertices of p , the paths $(1_G, w_1^{-1})$ and $(1_G, w_2)$ run inside Γ_χ ; as the χ -tracks of u_1 and u_2 are non-negative, the path $p_1 = (1_G, \tilde{u}_1)$ runs likewise inside the subgraph Γ_χ . Similarly, one sees that the path $(1_G, \tilde{u}_2)$ stays inside the subgraph Γ_χ . The facts that $x_1 \cdot u_1$ and $x_2 \cdot u_2$ represent the same element in G and that $\chi(x_1) = \chi(x_2)$ imply next that $\chi(\tilde{u}_1) = \chi(\tilde{u}_2)$ and so the path $(1_G, \tilde{u}_1 \cdot \tilde{u}_2^{-1})$ runs inside Γ_χ . Its endpoint is g ; indeed,

$$\eta_*(\tilde{u}_1 \cdot \tilde{u}_2^{-1}) = u_1 \cdot u_2^{-1} = x_1^{-1} \cdot x_2 = (h^{-1})^{-1} \cdot h^{-1}g = g.$$

All together together, we have shown that every element of $\ker \chi$ belongs to \mathcal{C}_1 . So Lemma C1.8 allows us to conclude that the subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected. As $\eta: \mathcal{X} \rightarrow G$ is an arbitrary generating system, the proof is complete \square

Notes

There are two noteworthy situations in which the previous lemma can be applied to every non-zero character of the group. The first one is treated in Corollary C2.5, the second one in Corollary C2.9.

COROLLARY C2.5 *Assume G is a group that contains no free submonoid of rank 2. Then $\Sigma^1(G) = S(G)$.*

If, in addition, G is finitely generated the members of the derived series

$$G' = [G, G], \quad G'', \quad G''', \dots$$

of G are finitely generated and every soluble quotient of G is polycyclic.

Proof. Given a non-zero character χ of G and two elements x_1, x_2 with $\chi(x_1) = \chi(x_2) > 0$, there exist, by hypothesis, two distinct words v'_1, v'_2 in the alphabet $\{x_1, x_2\}$ which represent the same element of G ; they will have the same length. If they begin with the same letter, let v be the largest common initial subword and let v_1 and v_2 be the remainders. Then v_1 and v_2 are distinct words which also represent the same element of G and start, one with x_1 , the other with x_2 . By exchanging v_1 and v_2 , if need be, we may arrange that v_1 begins with x_1 and v_2 with x_2 . Then v_1 has the form $x_1 \cdot u_1$ and v_2 the form $x_2 \cdot u_2$. Since u_1, u_2 have non-negative χ -tracks (for they are positive words) their χ -tracks satisfy requirement (C2.2) and so $[\chi] \in \Sigma^1(G)$ by Lemma C2.4.

Assume now that H is a finitely generated subgroup of G . Then H contains no a free submonoid of rank 2, whence $\Sigma^1(H) = S(H)$ by the part already proved. Theorem A4.1 then implies that the derived group H' of H is finitely generated. If the group G itself is finitely generated, this argument can be applied successively to the terms $G^{(m)}$ of the derived series of G and shows that each of them is finitely generated. It follows, in particular, that every quotient group $G^{(m)}/G^{(m+1)}$ is finitely generated abelian and that every quotient $G/G^{(m+1)}$ is polycyclic. \square

NOTE C2.6 Corollary C2.5 is due to G. Levitt (see [Lev87, p. 659, Remarques]). In [LMR95], the addendum to the corollary is proved by a direct argument (see Corollary 3 on page 1421.)

EXAMPLES C2.7 a) There are two classes of groups which, for obvious reasons, do not contain non-abelian free submonoids: the class of *torsion groups* and the class of *abelian groups*. More surprisingly, nilpotent and thus locally nilpotent groups contain no non-abelian free submonoids, either. A simple proof of this fact is given in A. Shalev's paper [Sha91] (see part (iv) of Proposition 7.1). To describe this proof, we introduce two sequences $n \mapsto u_n$ and $n \mapsto v_n$ of positive words on the alphabet $\{x, y\}$, defined as follows:

$$\begin{aligned} u_1 &= xy, & v_1 &= yx, \\ u_2 &= u_1 v_1 = xy^2x, & v_1 &= v_1 u_1 = yx^2y, \end{aligned}$$

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and, in general, $u_{n+1} = u_n v_n$ and $v_{n+1} = v_n u_n$. Notice that $u_n(x, y)$ and $v_n(x, y)$ are distinct words (of length 2^n).

Suppose now x and y are elements in a group H such that the relation $u_n(x, y) = v_n(x, y)$ holds modulo the centre $\zeta(H)$ of H . There exists then a central element $z \in H$ with $v_n(x, y) = u_n(x, y) \cdot z$ and so

$$u_{n+1}(x, y) = u_n v_n = u_n \cdot u_n z = u_n z \cdot u_n = v_n u_n = v_{n+1}(x, y).$$

It follows inductively that the relation $u_n = v_n$ is a law in every nilpotent group of class at most n .

b) Consider next an *extension* G of a locally nilpotent normal subgroup N by a torsion group G/N . Given g and h in G there exists then a positive integer m such that $x = g^m$, $y = h^m$ lie in N ; by hypothesis, the elements x, y generate a nilpotent subgroup, say of class n , whence part a) allows us to conclude that the equation $u_n(g^m, h^m) = v_n(g^m, h^m)$ holds in G . This equation testifies that x and y do not generate a free submonoid of G .

A concrete example of such a group G is described on page 1420 of [LMR95]: let F be a non-abelian free group and $R \triangleleft F$ a normal subgroup such that F/R is a torsion group. Then $G = F/R'$ is abelian-by-torsion. Moreover, the canonical projection $F \twoheadrightarrow F/R'$ induces an isomorphism $F_{\text{ab}} \xrightarrow{\sim} G_{\text{ab}}$. It shows that G_{ab} is free abelian of positive rank and implies that G is torsion-free.

c) Consider finally a finitely generated *soluble* group G that does not contain a non-abelian free submonoid. By a result of J. M. Rosenblatt G is then nilpotent-by-finite ([Ros74], see also [LMR95, Theorem 1]). Conversely, by the previous example b) every nilpotent-by-finite group contains no non-abelian free submonoid.

C2.1c Illustration 2: Engel groups

Engel groups are generalizations of nilpotent groups. They are defined like this (cf. [Rob96, pp. 371–376]):

DEFINITION C2.8 Let x, y be elements of a group and $n \geq 1$ a natural number. The iterated commutators $n \mapsto c_n(x, y)$ are defined inductively like this:

$$c_1(x, y) = [x, y] = xyx^{-1}y^{-1}, \quad (\text{C2.3})$$

$$c_{n+1}(x, y) = [c_n(x, y), y] = c_n(x, y) \cdot y \cdot c_n(x, y)^{-1} \cdot y^{-1} \text{ for } n \geq 1. \quad (\text{C2.4})$$

A group is called an *Engel group* if there exists for every couple (x, y) of elements in G a natural number $k \geq 1$ such that $c_k(x, y) = 1$.

The following result is the analogue of Corollary C2.5:

COROLLARY C2.9 If G an Engel group then $\Sigma^1(G) = S(G)$. If G is finitely generated so are the members of its derived series.

Notes

Proof. Given a non-zero character χ of G , let x_1, x_2 two elements of G satisfying the condition $\chi(x_1) = \chi(x_2)$. By hypothesis, there exists then an index $m > 1$ so that the iterated commutator $c_m(x_1, x_2)$ vanishes. If $m = 1$, the relation $x_1 \cdot x_2 = x_2 \cdot x_1$ is then valid and shows that the hypothesis of Lemma C2.4 is satisfied for the triple (χ, x_1, x_2) .

Assume next that $m > 1$ and set $n = m - 1$ as well as $x = x_1, y = x_2$. The relation

$$c_n(x, y) \cdot y = y \cdot c_n(x, y) \quad (\text{C2.5})$$

then holds. In general, the inductive definition of $c_n(x, y)$ does not produce a freely reduced word; but it is easy to describe the free reduction of $c_n(x, y)$ by an inductive scheme.

LEMMA C2.10 *Let $n \mapsto w_n$ be the sequence of words in $\{x, x^{-1}, y, y^{-1}\}$ defined by*

$$w_1 = 1 \quad \text{and} \quad w_{n+1} = w_n \cdot x^{-1}yx \cdot w_n^{-1} \cdot y^{-1}. \quad (\text{C2.6})$$

Then each word w_n is freely reduced; if $n > 1$, it starts with x^{-1} and ends with y^{-1} . Moreover, for each $n \geq 1$, the word $c'_n = xy \cdot w_n \cdot x^{-1}y^{-1}$ is freely reduced and is the free reduction of $c_n(x, y)$.

The lemma can be verified by a straightforward induction.

Let's now go back to the relation (C2.5). If one replaces in it the commutator $c_n(x, y)$ by its free reduction $c'_n = xy \cdot w_n \cdot x^{-1}y^{-1}$ and discards the subword $y^{-1}y$ at the end of the left term, one arrives at the relation

$$x \cdot u = y \cdot v \quad \text{with } u = yw_nx^{-1} \text{ and } v = x(yw_nx^{-1})y^{-1}.$$

Since $\chi(x) = \chi(y) > 0$, the inductive definition of w_n allows one to deduce that the χ -track of u is non-negative, whence so it that of v . The hypothesis of Lemma C2.4 holds therefore also for indices $m > 1$.

The additional claim then follows as in the proof of Corollary C2.5. \square

NOTE C2.11 The addendum to Corollary C2.9 is due to K. W. Gruenberg ([Gru53, Theorem 2]).

EXAMPLES C2.12 Every nilpotent group, hence every locally nilpotent group, is an Engel group. Conversely, every *finite* Engel group is nilpotent ([Zor36]), as is every *finitely generated soluble* Engel group ([Gru53, Theorem 1]).

On comparing these results with those stated in examples C2.7 a) and c), one sees that, for finite groups and for finitely generated soluble groups, the hypothesis of being an Engel group is more stringent than that of being a group with no free submonoids of rank 2.

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C2.1d Illustration 3: amalgamated free products and HNN-extensions

In the preceding sections C2.1b and C2.1c, we considered situations which allow one to conclude that every point $[\chi]$ lies in $\Sigma^1(G)$. In this section, we turn to constructions that allow one to infer that certain points are outside of Σ^1 . The constructions will amalgamated free products and HNN-extensions. One has:

PROPOSITION C2.13 (i) *Let G be an amalgamated free product $G_1 \star_A G_2$ and $\chi: G \rightarrow \mathbb{R}$ a non-zero character. If A is distinct from both G_1 and G_2 and χ vanishes on A , then $[\chi] \notin \Sigma^1(G)$.*

(ii) *Let G be an HNN-extension $\langle B, t \mid t \cdot s \cdot t^{-1} = \sigma(s) \text{ for } s \in S \rangle$ with base group B , associated subgroups S, T and isomorphism $\sigma: S \xrightarrow{\sim} T$. If the character χ vanishes on S then $[\chi] \notin \Sigma^1(G)$, except, possibly, if $\chi(t) < 0$ and $S = B$ or $\chi(t) > 0$ and $T = B$.*

Proof. (i) Assume first that χ is non-zero on G_1 . Choose elements $g_1 \in G_1$ with $\chi(g_1) > 0$ and $g_2 \in G_2 \setminus A$ with $\chi(g_2) \geq 0$, and set $g = g_1^{-1}g_2g_1$. Then $\chi(g_1^{-1}) < 0$, but $\chi(g) \geq 0$. Consider now the generating set $\mathcal{X} = G_1 \cup G_2$; let $\Gamma = \Gamma(G, \mathcal{X})$ be the associated Cayley graph and Γ_χ its subgraph corresponding to χ .

We claim that Γ_χ is not connected. Indeed, the word $w_0 = (g_1^{-1}, g_2, g_1)$ is in reduced form and it describes a path $p = (1, w_0)$ in Γ that start and ends in Γ_χ , but leaves Γ_χ . If w is another word in the alphabet $\mathcal{X}^\pm = G_1 \cup G_2$ that defines the element $g \in G$, then w can be reduced to a word w' in normal form. In this reduction the value of the function $v_\chi: W(\mathcal{X}^\pm) \rightarrow \mathbb{R}$ will not decrease. By the *Reduced-Form Theorem* (see, e. g., [Coh89, Chapt.1, Thm. 26]), w' has the form (h_1, h_2, h_3) with $h_1 \in G_1 \setminus A$, $h_2 \in G_2 \setminus A$, $h_3 \in G_1 \setminus A$ and $h_1 \in g_1^{-1}A$. But if so, the assumption that χ vanishes on A implies that $\chi(h_1) = \chi(g_1^{-1}) < 0$ and thus the subgraph Γ_χ is not connected.

If χ vanishes on G_1 , exchange the rôles of G_1 and G_2 in the preceding argument and infer, as before, that $[\chi] \notin \Sigma^1(G)$.

(ii) The proof will be similar to the preceding one; it is based on the *Reduced Form Theorem* for HNN-extensions (see, for instance, [Coh89, p. 36, Thm. 37]).

Three cases will be distinguished; in each of them we choose $\mathcal{X} = B \cup \{t\}$ as generating set. Assume first that χ does not vanish on B . Choose an element $b \in B$ with $\chi(b) > 0$; notice that $b \notin S \cup T$. Consider the word $w_0 = (b^{-1}, t^\varepsilon, b)$ where the sign ε is to be chosen so that $\chi(y^\varepsilon) \geq 0$. Then w_0 is in reduced form; as χ is negative on its first letter it follows, as in part (i), that Γ_χ is not connected.

Assume now that $\chi(B) = \{0\}$. If $\chi(t) < 0$ and $S \neq B$, pick $b \in B \setminus S$ and consider the word $w_0 = (t, b, t^{-1})$. It is in reduced form and allows one to deduce that Γ_χ is not connected. If, on the other hand, $\chi(t) > 0$ and $T \neq B$, find $b \in B \setminus T$, and set $w_0 = (t^{-1}, b, t)$. This word is reduced and reveals that Γ_χ is not connected. \square

REMARKS C2.14 a) Part (i) of Proposition C2.13 generalizes the conclusion stated in Example 3 of section A2.1a.

b) The exceptional cases listed in statement (ii) of the preceding proposition deserve a comment. Assume that $\chi(t) > 0$. If $[\chi] \in \Sigma^1(G)$ the proposition implies that T coincides with B . This conclusion brings to light that Proposition B3.7 continues to be valid if the base group B is allowed to be infinitely generated.

Suppose now that $T = B$. If B , and hence G , are *finitely generated*, then $[\chi] \in \Sigma^1(G)$ by Proposition A3.4. If, however, B is infinitely generated the point $[\chi]$ may lie outside of $\Sigma^1(G)$, but it can also be contained in it. One reason is this: the assumption that $T = B$ is infinitely generated does not exclude the case where $B = N = \ker \chi$. Then each of the three situations

$$\text{a) } [\chi] \in \Sigma^1(G) \text{ and } [-\chi] \notin \Sigma^1(G), \quad \text{b) } [\chi] \notin \Sigma^1(G) \text{ and } [-\chi] \in \Sigma^1(G),$$

and $[\chi] \notin \Sigma^1(G)$ and $[-\chi] \notin \Sigma^1(G)$ are possible.

In the previous example the group G may be chosen to be finitely generated. Here is a different type of example where G will be infinitely generated. Let B be a strictly ascending union $\bigcup_{j \in \mathbb{N}} B_j$ of finitely generated subgroups B_j and assume that $B_j \subseteq tB_jt^{-1}$ for every index $j \in \mathbb{N}$. Then $G = \text{gp}(B \cup \{t\})$ is an ascending union of finitely generated subgroups $G_j = \text{gp}(B_j \cup \{t\})$, and $[\chi|_{G_j}] \in \Sigma^1(G_j)$ for every index j . Proposition C2.50 thus allows us to conclude that $[\chi] \in \Sigma^1(G)$.

Particular instances of the previous type of example are certain extensions of locally finite groups by infinite cyclic groups. They reveal that the conclusion of Lemma B3.21 need not be true if the group G is not finitely generated.

We conclude with an application of Proposition C2.13.

COROLLARY C2.15 *If G is a finitely generated group with infinitely many ends then $\Sigma^1(G)$ is empty.*

Proof. By Stallings' structure theorem (see [Sta71, 5.A.9]), a finitely generated group G with infinitely many ends is either a non-trivial free product $G_1 \star_A G_2$ with finite amalgamated group A , properly contained in both factors and of index greater than 2 in at least one factor, or an HNN-extension

$$\langle B, t; t \cdot s \cdot t^{-1} = \sigma(s) \text{ for } s \in S \rangle$$

with finite associated subgroups S, T , both properly contained in B . As every character vanishes on a finite subgroup, Proposition C2.13 allows us to see that no character represents a point of $\Sigma^1(G)$. \square

C2.1e A glimpse of the invariants Σ^M and Σ^B

In this final part of section C2.1 I shall describe two constructions that underlie the proofs of the equivalence of Σ^1 with the invariants propounded by Gaël Meigniez in [Mei87], [Mei90] and by Ken Brown in [Bro87b].

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Let G be a group and $\chi: G \rightarrow \mathbb{R}$ a non-zero character. Given a generating system $\eta: \mathcal{X} \rightarrow G$ of G we are interested in finding out whether the subgraph $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$ of the Cayley graph $\Gamma(G, \mathcal{X})$ is connected. If the subgraph Γ_χ is *not* connected it has infinitely many components.⁶ In a case where one suspects that Γ_χ is not connected it might therefore be more profitable to consider subsets of connected components rather than individual components, and try to show that there exists such a subset whose union is neither empty nor all of Γ_χ . One is thus led to consider subsets of G with properties resembling those of the vertex sets of unions of connected components of G_χ . Gaël Meigniez has found a framework in which such subsets can conveniently be discussed. One benefit of his approach is the fact that the generating system \mathcal{X} does not enter into the calculations.

Ken Brown has detected another avenue to the problem of finding out whether the subgraph Γ_χ is connected; it involves an action of G . Let $C = C(\Gamma_\chi)$ denote the set of all connected components of Γ_χ . The group G acts on the Cayley graph $\Gamma(G, \mathcal{X})$ by left translations, but this action does not induce one on C , as C is only invariant under $N = \ker \chi$. Now in the applications of Σ^1 one typically has some information on G and aims at using it to deduce properties of N . So one should try to enlarge the N -set $C(\Gamma_\chi)$ to a G -set, say T . This goal can be reached in different ways, one solution being the following one.

Let $T = T(G, \mathcal{X}, \chi)$ be the set of all connected components of the subgraphs $\Gamma(G, \mathcal{X})_\chi^{[b, \infty)}$ with b ranging over $\text{im } \chi \subseteq \mathbb{R}$. Then T carries an obvious G -action and it is equipped with additional structure: the homomorphism χ , for instance, induces a map $\lambda: T \rightarrow \mathbb{R}$ that sends a component of $\Gamma(G, \mathcal{X})_\chi^{[b, \infty)}$ to b . This map λ is G -equivariant; it is injective if, and only if, Γ_χ is connected or, expressed in a more suggestive, geometric language, if T is a “line”. One crucial observation is now this: every element $g \in G$ with $\chi(g) \neq 0$ gives rise to a subset A_g that is invariant under $\text{gp}(g)$ and enjoys properties similar to those of the axis of a hyperbolic element of a group acting on an \mathbb{R} -tree (see Lemma C2.29 for details). Using the axes A_g of suitably chosen elements g one can try to deduce that T is a line; if one succeeds, one has shown that Γ_χ is connected.

The preceding paragraph indicates how the construction of T can help one in proving that a given point $[\chi]$ belongs to $\Sigma^1(G)$. But more is true. The additional structure of the G -set T turns it into a kind of tree, to be called the *measured tree* associated to G , \mathcal{X} and χ . One has also a notion of an abstract measured tree which comprises, in particular, certain quotients of T . An astonishing fact is now this: it there exists, given a group G and a non-zero character χ of G , an abstract measured tree T' which is *not* a line there exists a generating system \mathcal{X} of G so that the measured tree $T(G, \mathcal{X}, \chi)$ associated to G , \mathcal{X} and χ is not a line (see Proposition C2.46), whence $[\chi] \notin \Sigma^1(G)$. So the approach via actions on tree-like structures also provides a tool for showing that a point $[\chi]$ does *not* belong to $\Sigma^1(G)$.

The approach has a third merit. The notion of an abstract measured tree T' does not involve the choice of a generating system; in particular, an abstract

⁶See the proof of Corollary A2.10.

measured tree T' associated to G and χ gives rise to an abstract measured tree for every subgroup G_1 of G with $\chi(G_1) \neq \{0\}$. This fact allows one to use the hypothesis that $[\chi|_{G_1}] \in \Sigma^1(G_1)$ in proving that $[\chi] \in \Sigma^1(G)$. In this way one can show, for instance, that the statement of Proposition B1.15, dealing with joins of subgroups, carries over to infinitely generated groups (see Proposition C2.50).

C2.2 Ends in the direction of a character (G. Meigniez)

Gaël Meigniez introduces in [Mei87] (cf. [Mei88] and [Mei90]) an invariant Σ^M whose definition is phrased in terms reminiscent of the number of ends of a space. This section begins therefore with a reminder of this notion. Then follow the definition of the invariant Σ^M and the proof of the equality of Σ^M and Σ^1 (see Theorem C2.19).

C2.2a Number of ends of a topological space — a reminder

Let X be a locally finite, connected simplicial complex. For each finite subcomplex K , the number of connected components of $X \setminus K$ is finite; let $n(K)$ denote the number of the infinite ones. The number of ends $e(X)$ of the space X is then defined to be the supremum

$$\sup\{n(K) \mid K \subset X \text{ finite subcomplex}\}. \quad (\text{C2.7})$$

This number is either a positive integer or ∞ . (See, e. g., [SW79, Section 5] for further information.)

The preceding definition applies, in particular, to the Cayley graph $\Gamma = \Gamma(G, \mathcal{X})$ of an infinite group with a finite generating system $\eta: \mathcal{X} \rightarrow G$. The infinite components of a subset $\Gamma \setminus K$ are then infinite connected subgraphs $S \subset \Gamma \setminus K$ whose boundaries $\delta(S) = \delta_{\mathcal{X}}(S)$ are finite. Here a vertex v is said to lie in the boundary of S if it is the endpoint of an edge having one extremity in S and the other in $G \setminus S$; more precisely,

$$\delta_{\mathcal{X}}(S) = \bigcup_{y \in \mathcal{X} \cup \mathcal{X}^{-1}} (S \cdot y \setminus S) \cup (S \setminus S \cdot y). \quad (\text{C2.8})$$

The concept of the *number of ends* of a topological space goes back to [Hop44]. In this paper, Heinz Hopf establishes, inter alia, the following results:

- (i) let $X = \tilde{Y}$ be the universal covering of a finite, connected simplicial complex Y with infinite fundamental group $G = \pi_1(Y)$. Then the number of ends of X is 1, 2 or ∞ ([Hop44, p. 91, Satz I]);
- (ii) the number of ends of a Cayley graph $\Gamma = \Gamma(G, \mathcal{X})$ of an infinite, finitely generated group does not depend on the choice of the finite generating system $\eta: \mathcal{X} \rightarrow G$ and is therefore an invariant of the group G ([Hop44, p. 96, Art. 16]).

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Hopf establishes claims (i) and (ii) by topological arguments and then raises the question of finding a purely algebraic theory of the number of ends of a group. Hans Freudenthal proposes such a theory in [Fre45].

Here are some elements of his theory; they will be important in the sequel. Assume G is a finitely generated, infinite group, $\mathcal{X} \hookrightarrow G$ is a finite generating set of G , and $\Gamma = \Gamma(G, \mathcal{X})$ the corresponding Cayley graph. Consider a subset $A \subseteq G$. It gives rise to a (full) subgraph $\Gamma|_A$ of Γ ; let $\delta_{\mathcal{X}}(A)$ denote its boundary in Γ . Then $\delta_{\mathcal{X}}(A)$ is finite if, and only if, the set

$$\delta_h(A) = (Ah \setminus A) \cup (A \setminus Ah) \quad (\text{C2.9})$$

is finite for every $h \in G$. It follows that the answer to the question whether a set A has a finite boundary $\delta_{\mathcal{X}}(A)$ with respect to a finite generating set \mathcal{X} of G does not depend on the choice of \mathcal{X} .

Let us apply this fact to an *infinite connected component* \mathcal{C} of the subgraph of Γ spanned by the complement $G \setminus \mathcal{F}$ of a finite subset \mathcal{F} . The boundary of \mathcal{C} is then finite. Assume now \mathcal{X}' is another finite generating set of G . Then $A = V(\mathcal{C})$ induces a subgraph $\Gamma|_A$ of $\Gamma(G, \mathcal{X}')$. This subgraph may not be connected, but it has a finite boundary and so it contains at least one infinite connected component. As the rôles of $\Gamma(G, \mathcal{X})$ and $\Gamma(G, \mathcal{X}')$ can be interchanged, the previous reasoning constitutes an algebraic proof of Hopf's assertion (ii); this proof yields, in addition, a characterization of groups with one end: *a finitely generated group G has one end if, and only if, the following implication holds for every subset $A \subset G$:*

$$\delta_h(A) \text{ is finite for every } h \in G \implies A \text{ or } G \setminus A \text{ is finite.} \quad (\text{C2.10})$$

This algebraic characterization of a finitely generated group with one end will be the starting point of the next section.

C2.2b Number of ends in the direction of a character

Let G be a group and $\chi: G \rightarrow \mathbb{R}$ a non-zero character. The definition of a group with one end in the direction of χ is obtained from characterization (C2.10) of a finitely generated group with one end by replacing in the latter characterization the word *finite* by the phrase *with χ -values that are bounded from above*.

DEFINITION C2.16 G has *one end in the direction of χ* if the following implication holds for every subset $A \subset G$:

$$\begin{aligned} \chi(\delta_h(A)) \text{ is bounded from above for every } h \in G \\ \Downarrow \\ \chi(A) \text{ or } \chi(G \setminus A) \text{ is bounded from above.} \end{aligned}$$

A glance at definition C2.16 shows that if G has one end in the direction of χ it has one end in the direction of every positive multiple of χ . The definition allows one therefore to single out a subset $\Sigma^M(G)$ of the sphere $S(G)$, namely

$$\Sigma^M(G) = \{[\chi] \in S(G) \mid G \text{ has one end in the direction of } \chi\}. \quad (\text{C2.11})$$

Notes

We shall prove that Σ^M coincides with Σ^1 . Prior to establishing this result, we pause for some historical comments.

NOTES C2.17 a) The concept of *ends of a group G in the direction of a character* is due to Gaël Meigniez (see [Mei87, §1]). In [Mei87], a number of results involving this concept are established, in particular the following two: firstly, if G is finitely generated, the subsets $\Sigma^M(G)$ and $\Sigma^1(G)$ coincide (Corollaire 5); secondly, the analogue of Proposition C2.13 for the invariant Σ^M (see Theorems 4.1 and 4.2).

b) The concept of ends of a group in a given direction is also of interest for other types of “characters”. This novel type of generalization is discussed in Gaël Meigniez’ thesis [Mei88] and in his article [Mei90]; it has been taken up by Robert Bieri in [Bie93].

C2.2c Equivalence of Σ^M and Σ^1

In this section, it is shown that a group G has one end in the direction of a character if, and only if, that character represents a point of $\Sigma^1(G)$. We begin with

LEMMA C2.18 *Let G be a group with generating system $\eta: \mathcal{X} \rightarrow G$. Consider a subset A of G . If $\chi(\delta_x(A))$ is bounded from above for each $x \in \mathcal{X}$ then $\chi(\delta_h(A))$ is bounded from above for every $h \in G$.*

Proof. Given an element $h \in G$, write it as a product $y_1 \cdot y_2 \cdots y_\ell$ with factors $y_j = x_j^{\varepsilon_j} \in \mathcal{X} \cup \mathcal{X}^{-1}$. By hypothesis, the set $\chi(\delta_x(A))$ is bounded from above for each $x \in \mathcal{X}$. The calculation

$$\begin{aligned} \delta_{x^{-1}}(A) &= (Ax^{-1} \setminus A) \cup (A \setminus Ax^{-1}) \\ &= (A \setminus Ax) \cdot x^{-1} \cup (Ax \setminus A) \cdot x^{-1} = \delta_x(A) \cdot x^{-1} \end{aligned}$$

then shows that each $\chi(\delta_{x^{-1}}(A))$ is likewise bounded from above. It follows, in particular, that each set $\chi(\delta_{y_i}(A))$ is bounded from above; in particular, $\chi(\delta_{y_1}(A))$. Write h as a product $h' \cdot y$ with $h' = y_1 \cdot y_2 \cdots y_{\ell-1}$ and $y = y_\ell$ and assume, inductively, that $\chi(\delta_{h'}(A))$ is bounded from above. The computation

$$\begin{aligned} \delta_{h'y}(A) &= (Ah'y \setminus A) \cup (A \setminus Ah'y) \\ &\subseteq (Ah'y \setminus Ay) \cup (Ay \setminus A) \cup (A \setminus Ay) \cup (Ay \setminus Ah'y) \\ &= (Ah' \setminus A) \cdot y \cup \delta_y(A) \cup (A \setminus Ah') \cdot y \\ &= \delta_y(A) \cup \delta_{h'}(A) \cdot y \end{aligned}$$

then shows that $\chi(\delta_h(A))$ is bounded from above. □

THEOREM C2.19 *The invariants $\Sigma^M(G)$ and $\Sigma^1(G)$ coincide for every group G .*

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Proof. Let $\chi: G \rightarrow \mathbb{R}$ be a non-zero character. If $[\chi] \notin \Sigma^1(G)$, there exists a generating system $\eta: \mathcal{X} \rightarrow G$ for which the subgraph Γ_χ is not connected. We claim that every path component \mathcal{C} of Γ_χ has arbitrary large χ -values, but boundaries which are bounded from above. Set $A = V(\mathcal{C})$. Clearly $\chi(A)$ is not bounded from above. Consider now a boundary $\delta_h(A)$ with $h \in G$. Assume first that $h = x \in \mathcal{X}$. If g is an element of

$$\delta_x(A) = (Ax \setminus A) \cup (A \setminus Ax)$$

then either $g \notin A$ but $gx^{-1} \in A$, or $g \in A$ but $gx^{-1} \notin A$. In the first case, $\chi(g) < 0$; in the other case, $\chi(x) > 0$ and $\chi(g) < \chi(x)$. In both cases, $\chi(g)$ is bounded from above by $|\chi(x)|$. It follows that $\chi(\delta_x(A))$ is bounded from above for each $x \in \mathcal{X}$ and so Lemma C2.18 implies that $\chi(\delta_h(A))$ is bounded from above for every $h \in G$. The fact that $\Gamma(G, \mathcal{X})_\chi$ has several (infinite) path components allows us, finally, to deduce that *G has more than one end in the direction of χ .*

Conversely, assume that $[\chi] \in \Sigma^1(G)$. Let A be a subset of G which has unbounded χ -values, but boundaries $\delta_h(A)$ whose χ -values are bounded from above. We aim at showing that $\chi(G \setminus A)$ is bounded from above. To reach this goal, we shall construct a generating set \mathcal{X} of G and then consider the subgraph $\Gamma(G, \mathcal{X})_\chi$; since $[\chi] \in \Sigma^1(G)$ this subgraph is connected. The construction of \mathcal{X} will imply that the vertex set of a translate of $\Gamma(G, \mathcal{X})_\chi$ is contained in A ; this, in turn, will allow us to conclude that the χ -values of $G \setminus A$ are bounded from above.

Choose an element $t \in G$ with $\chi(t) > 0$ and let β_t denote an upper bound of $\chi(\delta_t(A))$. Since χ is not bounded on A , there exists a vertex $g_* \in A$ with $\chi(g_*) \geq \beta_t$. The ray $\ell \mapsto g_* \cdot t^\ell$ belongs then entirely to A ; here $\ell \geq 0$. Consider now an element $h \in G_\chi \setminus \{t\}$. Since $\chi(\delta_h(A))$ is bounded from above there exists a positive integer $\ell(h)$ so that the path

$$p_h = (g_*, t^{\ell(h)} h t^{-\ell(h)})$$

has all its vertices in A . (Recall that $\chi(\delta_t(A))$ is bounded from above by β_t and that $\chi(g_*) \geq \beta_t$.) The set

$$\mathcal{X} = \{t\} \cup \left\{ x_h = t^{\ell(h)} h t^{-\ell(h)} \mid h \in G_\chi \setminus \{t\} \right\}.$$

obviously generates G ; since $[\chi] \in \Sigma^1(G)$, the subgraph $\Gamma(G, \mathcal{X})_\chi$ is therefore connected. For each $g \in G_\chi$ there thus exists an \mathcal{X}^\pm -word w that represents g and has non-negative χ values on all its initial segments; let $x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}$ be the spelling of w . Each generator x_j is a conjugate, say

$$x_j = t^{\ell_j} h_j t^{-\ell_j},$$

of an element $h_j \in G_\chi$ (if $x_j = t$ set $\ell_j = 0$). The word w gives rise to the sequence

$$s = (t^{\ell_1}, h_1^{\varepsilon_1}, t^{-\ell_1}, t^{\ell_2}, h_2^{\varepsilon_2}, t^{-\ell_2}, \dots, t^{\ell_k}, h_k^{\varepsilon_k}, t^{-\ell_k}).$$

This sequence, in turn, leads to a path $p_g = (g_*, s)$ that starts at g_* and ends at g_*g ; we assert that all of its vertices lie in A .

Notes

For the origin and the vertices of the subsequence $s_1 = (t^{\ell_1}, h_1^{\varepsilon_1}, t^{-\ell_1})$ this holds by the choice of g_* and the integer ℓ_1 ; it follows that the end point $g_1 = g_* x_1^{\varepsilon_1}$ of the subpath corresponding to $s_1 = x_1^{\varepsilon_1}$ lies in A . Since the χ -value of the end point g_1 is at least as large as that of g_* , the choice of h_2 then implies that the vertices of the second subsequence $s_2 = (t^{\ell_2}, h_2^{\varepsilon_2}, t^{-\ell_2})$ lie in A , and that the χ -value of the end point g_2 of the subpath corresponding to $x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2}$ is at least as large as $\chi(g_*) \geq \beta_t$. And so on and so forth. It follows that $g_k = g_* g$, the end point of the entire path, lies in A .

The subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected; hence so the translated subgraph

$$G_\chi^{[\chi(g_*), \infty)} = \{g \in G \mid \chi(g) \geq \chi(g_*)\}.$$

The preceding argument thus implies that the vertex set of this entire subgraph is contained in A and so $\chi(G \setminus A)$ is bounded from above, namely by $\chi(g_*)$. \square

NOTE C2.20 The above theorem is due to G. Meigniez: the inclusion $\Sigma^1(G)^c \subseteq \Sigma^M(G)^c$ is established in [Mei87] for finitely generated groups and the argument given there generalizes easily to arbitrary groups. The proof of inclusion $\Sigma^1(G) \subseteq \Sigma^M(G)$ is taken from a letter which G. Meigniez posted to R. Strebel in 1991.

C2.3 Tree-like structures associated to a Cayley graph

The discussion of Ken Brown's approach will be in two parts. In this section, we first explain how a Cayley graph $\Gamma(G, \mathcal{X})$ and the choice of a point $[\chi] \in S(G)$ give rise to a tree-like structure $T = T(G, \mathcal{X}, \chi)$ and a canonical action of G on T . We next show by way of examples that the action of G on T can help one in proving that $\Gamma(G, \mathcal{X})_\chi$ is connected. In sections C2.4 and C2.5, we go one step further and prove that membership in Σ^1 can be characterized in terms of certain actions on \mathbb{R} -trees or on similar tree like structures.

C2.3a The construction of $T(G, \mathcal{X}, \chi)$

Let G denote a group, $\eta: \mathcal{X} \rightarrow G$ a generating system and $\chi: G \rightarrow \mathbb{R}$ a non-zero character. Set $\Gamma_\chi = \Gamma(G, \mathcal{X})_\chi$. The goal is to construct a tree-like structure T , equipped with a G -action, *that is a line if, and only if, Γ_χ is connected*.

We begin with

DEFINITION C2.21 Define the set T , the relation \leq and the function $\lambda: T \rightarrow \mathbb{R}$ as follows:

$$T = \bigcup_{r \in \text{im } \chi} \{\mathcal{C} \mid \mathcal{C} \text{ connected component of } \Gamma(G, \mathcal{X})_\chi^{[r, \infty)}\}, \quad (\text{C2.12})$$

$$\leq : \text{relation defined by } \mathcal{C} \leq \mathcal{C}' \iff \mathcal{C} \supseteq \mathcal{C}', \quad (\text{C2.13})$$

$$\lambda : \mathcal{C} \mapsto \inf\{\chi(h) \mid h \in V(\mathcal{C})\}. \quad (\text{C2.14})$$

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The triple (T, \leq, λ) enjoys some characteristic properties that will be enunciated in Lemma C2.23 below; in view of these properties, the structure (T, \leq, λ) will be called the *measured tree*⁷ associated to (G, \mathcal{X}, χ) . It will be denoted by $T(G, \mathcal{X}, \chi)$.

REMARK C2.22 Every point of T is a connected component of a subgraph

$$\Gamma_r = \Gamma(G, \mathcal{X})_\chi^{[r, \infty)}. \quad (\text{C2.15})$$

Since r is required to be the χ -value of an element $h \in G$, the component \mathcal{C}_1 , say, containing h has the property that $\chi(h) = \inf\{\chi(g_1) \mid g_1 \in V(\mathcal{C}_1)\}$. Actually, every component \mathcal{C} of Γ_r contains an element $h_{\mathcal{C}}$ with $\chi(h_{\mathcal{C}}) = \chi(h)$, the reason being that $\ker \chi$ acts transitively on the components of Γ_r (see Lemma A3.7). In the sequel, it will be convenient to call a component \mathcal{C}_h if h is a vertex of the component on which the infimum is attained.

The next result lists some basic properties of $T(G, \mathcal{X}, \chi)$; later on, these properties will be taken as the axioms of an (abstract) measured G -tree.

LEMMA C2.23 *For every group G , generating system $\eta: \mathcal{X} \rightarrow G$ and non-zero character χ , the associated measured tree $T(G, \mathcal{X}, \chi)$ satisfies the properties*

- (i) *the relation \leq is a partial order on T ;*
- (ii) *the relation \leq is directed; more precisely, for every pair a_1, a_2 in T there exists a point $a \in T$ with $a \leq a_1$ and $a \leq a_2$;*
- (iii) *for every $b \in T$ the set $(-\infty, b] = \{a \in T \mid a \leq b\}$ is linearly ordered and λ maps it bijectively and in an order-preserving way onto the subset $(-\infty, \lambda(b)] \cap \text{im } \lambda$ of the real line \mathbb{R} .*

Proof. Assertion (i) is clear. To prove (ii), assume a_1, a_2 are the components $\mathcal{C}_{g_1}, \mathcal{C}_{g_2}$, and choose a path p that connects g_1 with g_2 . If h is a vertex with minimal χ -value along p , the component $a = \mathcal{C}_h$ contains both vertices g_1 and g_2 , hence the components a_1 and a_2 , and so the relations $a \leq a_1$ and $a \leq a_2$ hold.

We are left with claim (iii). Assume $b = \mathcal{C}_g$ and consider components $a_1 = \mathcal{C}_{h_1}$ and $a_2 = \mathcal{C}_{h_2}$ which belong to the subset $(-\infty, b]$ of T . Then g is a vertex of both \mathcal{C}_{h_1} and \mathcal{C}_{h_2} . If $\lambda(\mathcal{C}_{h_1}) \leq \lambda(\mathcal{C}_{h_2})$ this implies that \mathcal{C}_{h_1} contains a path from h_1 to h_2 , hence the component \mathcal{C}_{h_2} and so $a_1 \leq a_2$. If $\lambda(\mathcal{C}_{h_2}) \leq \lambda(\mathcal{C}_{h_1})$ one sees similarly that $a_2 \leq a_1$. So the subset $(-\infty, b]$ of T is linearly ordered.

It remains to show that the restriction $\lambda_*: (-\infty, b] \rightarrow [-\infty, \lambda(b)] \cap \text{im } \chi$ of the function λ has the stated properties. The definitions of the order relation on T and of the function λ , given in (C2.13) and (C2.14), guarantee that λ_* preserves the order relations defined on the set $(-\infty, b] \subset T$ and on the interval $(-\infty, \lambda(b)]$.

⁷See Remark C2.24a for an explanation of this designation.

The injectivity of λ_* is a direct consequence of the above proof of claim (ii). The definition of λ_* and Remark C2.22 imply next that the value $\lambda(a)$ of a point a in the set $(\infty, b]$ lies in $\text{im } \chi$. So we are only left with proving that λ^* maps the subset $(-\infty, b]$ surjectively onto the interval $(-\infty, \lambda(b)] \cap \text{im } \chi$.

Suppose $b = \mathcal{C}_g$ and r' is an element of $\text{im } \chi$ with $r' \leq r = \lambda(\mathcal{C}_g)$. Then $r - r'$ is the χ -value of some element $h_0 \in G_\chi$. Write h_0 as a word w_0 in $\mathcal{X} \cup \mathcal{X}^{-1}$ and reorder the letters of w_0 so as to obtain a word w_1 with non-negative χ -track. The path (g, w_1^{-1}) will then lead from g to a element h' with $\chi(h') = \chi(g) - (r - r') = r'$; moreover, as the χ -track of w_1 is non-negative, this path runs in $\Gamma_{r'}$. The connected component $a' = \mathcal{C}_{h'}$ belongs therefore to the subset $(-\infty, b]$ of T and $\lambda(a') = r'$, as desired. \square

REMARKS C2.24 a) The chosen designation of the structure $T(G, \mathcal{X}, \chi)$ deserves some comment. The set T , equipped with the order relation \leq , is a partially ordered set or a poset. The fact that the subsets $(-\infty, b]$ sitting below a point b are linearly ordered turns the poset into a *forest in the sense of order theory*. As every pair of components $\mathcal{C}_1, \mathcal{C}_2$ in T has a lower bound, the forest is “connected” and hence a tree. The function λ , finally allows one to measure the length of intervals $[a, b] = \{c \in T \mid a \leq c \leq b\}$, a fact that explains the choice of the adjective “measured”.

b) Claim (iii) of the preceding lemma shows that every point $b \in T$ is the end point of a linearly ordered subset $(-\infty, b]$. This subset is uniquely determined by c and will be referred to as the *ray descending from c* . In view of (ii) two rays intersect in an infinite set; this set, however, need not be a ray as it may not possess a largest element.

c) The concept of a measured tree and the analysis of its structure — to be given in sections C2.3a and C2.3b — go back to sections II.1 and II.2 in the monograph [BS92], except for its name: in [BS92] it is called *measured rooted pre-tree* or *romp-tree*, for short. The new name arose out of a discussion of R. Strebel with Reinhard Diestel.

So far we know that the measured tree $T(G, \mathcal{X}, \chi)$ contains half-lines, namely the rays descending from its elements. As we shall see in the next section, it also contains *lines* in the sense of the following:

DEFINITION C2.25 A (non-empty) subset $L \subseteq T$ is called a *line* if it is linearly ordered, contains the ray of each of its elements and if $\sup\{\lambda(c) \mid c \in L\} = \infty$.

A ray of $T(G, \mathcal{X}, \chi)$ is determined by its largest element; a line by the choice of a linearly ordered sequence $c_0 < c_1 < c_2 < \dots$ with $\lim_{n \rightarrow \infty} \lambda(c_n) = +\infty$. The line defined by such a sequence is the union $L = \bigcup\{(-\infty, c_j] \mid j \in \mathbb{N}\}$.

REMARK C2.26 In general, it is difficult to get a good intuitive picture of a measured tree $T(G, \mathcal{X}, \chi)$, unless, of course, it is a line. If χ has rank 1, the situation improves: then $T(G, \mathcal{X}, \chi)$ is, in essence, a combinatorial tree (or \mathbb{Z} -tree) and can be described algebraically rather easily (see Claim (i) on p. 157).

C2.3b Canonical action on $T(G, \mathcal{X}, \chi)$

The group G acts on $T = T(G, \mathcal{X}, \chi)$; indeed, each component $\mathcal{C}_h \in T$ is a subgraph of the Cayley-graph $\Gamma(G, \mathcal{X})$ and the action of G on $\Gamma(G, \mathcal{X})$ induces an action of G on its set of subgraphs, in particular, on the subset of the components that are the elements of T . The following result amounts to say that G acts on $T(G, \mathcal{X}, \chi)$ by automorphisms.

LEMMA C2.27 *Let $T = T(G, \mathcal{X}, \chi)$ be the measured tree associated to G, \mathcal{X}, χ , and let $\mu: G \times T \rightarrow T$ denote the canonical action of G on T . Then G acts transitively on T and the following formulae hold for all $g, h \in G$ and all $a, b \in T$:*

$$g.\mathcal{C}_h = \mathcal{C}_{gh}, \quad (\text{C2.16})$$

$$a \leq b \Rightarrow g.a \leq g.b, \quad (\text{C2.17})$$

$$\lambda(g.a) = \chi(g) + \lambda(a). \quad (\text{C2.18})$$

Proof. The transitivity of the action and formulae (C2.16) and (C2.17) are immediate consequences of the definitions (see formula (C2.13)), while formula (C2.18) follows from the definition of λ and the computation

$$\lambda(g.\mathcal{C}_h) = \inf\{\chi(gh_1) \mid h_1 \in V(\mathcal{C}_h)\} = \chi(g) + \lambda(\mathcal{C}_h).$$

In the above calculation, one uses that $g.\mathcal{C}_h$ is the image of \mathcal{C}_h under the left translation $h \mapsto gh$. \square

The action of G on T allows one to assign to each $g \in G$ a subset A_g :

DEFINITION C2.28 For every $g \in G$, the subset

$$A_g = \{a \in T \mid a \leq g.a \text{ or } g.a \leq a\}. \quad (\text{C2.19})$$

is called the *characteristic subset* of g .

The basic properties of these characteristic subsets are described in

LEMMA C2.29 *For every $g \in G$ and measured tree $T = T(G, \mathcal{X}, \chi)$ the subset A_g has the following properties:*

- (i) A_g is not empty and contains with each $c \in A_g$ the ray descending from c .
- (ii) If $\chi(g) = 0$ then A_g coincides with the set $T^g = \{a \in T \mid g.a = a\}$ of points fixed by g .
- (iii) If $\chi(g) \neq 0$ then A_g is a line. It is the unique $\text{gp}(g)$ -invariant line of T and, for every point $a \in A_g$, the line A_g coincides with the union of the rays descending from the points of the orbit $\text{gp}(g).a$.

Notes

Proof. (i) Choose a point $b \in T$ and consider the intersection $I = (-\infty, b] \cap (-\infty, g.b]$. It is non-empty (the rays intersect because T is directed; see claim (ii) of Lemma C2.23). Let x be a point in I . Then $x \leq b$ and $x \leq g.b$ and so $g.x \leq g.b$; so x and $g.x$ lie both on the ray $(-\infty, g.b]$. As this ray which linearly ordered (by part (iii) of Lemma C2.23) either $x \leq g.x$ or $g.x \leq x$ must hold and so $x \in A_g$ by Definition C2.28. We next show that A_g contains, with every point $c \in T$, the ray descending from c . In this verification, we may and shall assume that $\chi(g) \leq 0$, for $A_g = A_{g^{-1}}$. Fix a point $c \in A_g$. Then $g.c \leq c$. If $a \leq c$ then $g.a \leq g.c$ by property (C2.17) and so $g.a \leq c$ by the transitivity of the relation \leq . So a and $g.a$ lie both on the linearly ordered ray descending from c . Since $\chi(g) \leq 0$, this shows that $g.a \leq a$ whence $a \in T$.

(ii) Clearly $T^g \subseteq A_g$. Conversely, if $a \in A_g$ then $a \leq ga$ or $ga \leq a$. The assumption that $\chi(g) = 0$ and statement (iii) in Lemma C2.23 then imply that $\lambda(a) = \lambda(g.a)$, whence $a \in T^g$.

(iii) By claim (i) the subset A_g is not empty; let a be one of its points. Then $a < g.a$ or $g.a < a$; in either case, $\{(g^m).a \mid m \in \mathbb{Z}\}$ is a totally ordered set that is not bounded from above and so the union $L_a = \bigcup \{(-\infty, g^m.a] \mid m \in \mathbb{Z}\}$ is a line; by claim (i), this line is contained in A_g . It follows that A_g contains all lines generated by the $\text{gp}(g)$ -orbits of points in A_g .

Let now a_1 and a_2 be two points of A_g and let L_1 and L_2 be the lines generated by the orbits $\text{gp}(g).a_1$ and $\text{gp}(g).a_2$, respectively. By claim (ii) in Lemma C2.23 there exists $a \in T$ with $a \leq a_1$ and $a \leq a_2$. It then follows that $L_1 \cap L_2$ contains the line generated by $\text{gp}(g).a$. Hence $L_1 \cap L_2$ is a line and so $L_1 = L_2$. This implies, in particular, that A_g is a line.

Consider, finally, a $\text{gp}(g)$ -invariant line L . If $a \in L$ then $g.a \in L$ by the $\text{gp}(g)$ -invariance of L ; as L is totally ordered, a and $g.a$ are comparable and so $a \in A_g$ by the very definition of A_g . Thus $L \subseteq A_g$. \square

DEFINITION C2.30 Let G be a group, $\eta: \mathcal{X} \rightarrow G$ a generating system, $\chi: G \rightarrow \mathbb{R}$ a non-zero character and $T = T(G; \mathcal{X}, \chi)$ the associated measured tree. An element $g \in G$ will be called *elliptic* if it lies in the kernel of χ , and otherwise *hyperbolic*. By the previous lemma, the characteristic subset A_g of a hyperbolic element is a line; it will be referred as the *axis* of g .

C2.3c A first application

In sections C2.3a and C2.3b, we saw that every triple $(G, \eta: \mathcal{X} \rightarrow G, \chi)$ leads to a measured tree $T(G, \mathcal{X}, \chi)$ that carries a canonical G -action. In this section, we illustrate how the properties of $T(G, \mathcal{X}, \chi)$ — stated in lemmata C2.23, C2.27 and C2.29 — lead to a simple proof of an attractive result established earlier by different techniques. The result deals with a group generated by a set \mathcal{X} having the property that many pairs of elements x_1, x_2 in \mathcal{X} commute.

We begin with a simple remark on the characteristic sets of commuting elements.

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LEMMA C2.31 Let $\eta: \mathcal{X} \rightarrow G$ be a generating system and $\chi: G \rightarrow \mathbb{R}$ a non-zero character of G . Consider the measured G -tree $T = T(G, \mathcal{X}, \chi)$.

Assume g and h are elements of G which commute and h is hyperbolic. Then the axis A_h of h is invariant under g . If, in addition, g is hyperbolic then $A_g = A_h$.

Proof. The definition (C2.19) of characteristic subsets implies that $A_h = A_{h^{-1}}$. We can, and shall, therefore assume that $\chi(h) > 0$. The calculation

$$g.A_h = g.\{a \in T \mid a \leq h.a\} = \{g.a \in T \mid g.a \leq (ghg^{-1}).(g.a)\} = A_{ghg^{-1}}$$

then shows that $g.A_h$ is the axis of ghg^{-1} . If g and h commute, the axis A_h is therefore invariant under g and so it is a $\text{gp}(g)$ -invariant line. Moreover, if g is hyperbolic, part (iii) Lemma C2.29 shows that A_h is the axis of g . \square

Now to the announced application.

EXAMPLE C2.32 Let G be a graph group, alias right angled Artin group, given by the (finite) combinatorial graph Δ , and let $\chi: G \rightarrow \mathbb{R}$ be a non-zero character. Let $\mathcal{L}(\chi)$ denote the living subgraph of χ ; by definition, this is the subgraph spanned by the set $\{x \in V(\Delta) \mid \chi(x) \neq 0\}$. According to Theorem B1.17, the character χ represents a point of $\Sigma^1(G)$ if, and only if, $\mathcal{L}(\chi)$ is connected and dominates Δ . In the proof, this condition is shown to be necessary by the construction of a suitable quotient group of G ; the sufficiency is deduced from a result on the invariant of a join of groups (namely Proposition B1.15).

The properties of the measured tree $T(G, \mathcal{X}, \chi)$ enable one to give a neater proof of the sufficiency. Pick a vertex x_1 be in $\mathcal{L}(\chi)$ and let $A = A_{x_1}$ be the axis of the corresponding generator. Consider another vertex x of $\mathcal{L}(\chi)$; since $\mathcal{L}(\chi)$ is assumed to be connected, there exists a path $(x_1, x_2, \dots, x_k = x)$ in $\mathcal{L}(\chi)$ from x_1 to x . The generators x_1 and x_2 commute; by Lemma C2.31 the axis A is invariant under x_2 ; in addition, it is the axis of the generator x_2 . Continuing in the indicated manner, one sees that A is a line which is invariant under every generator corresponding to a vertex of $\mathcal{L}(\chi)$. Now $\mathcal{L}(\chi)$ is also assumed to dominate Δ . So each generator outside $V(\mathcal{L}_\chi)$ commutes with a generator in $V(\mathcal{L}_\chi)$, whence the line A is invariant under all generators of G and so under G . As G acts transitively on $T(G, \mathcal{X}, \chi)$ this implies, in particular, that $\Gamma(G, V(\Delta))_\chi$ is connected.

REMARK C2.33 The above proof applies also to quotient groups of graph groups and it allows of some modifications. A striking modification is used in the proof of the main result of L. A. Orlandi-Korner's article [OK00].

C2.4 Characterization of Σ^1 in terms of actions on trees

In this section, Σ^1 will be described in terms of actions on trees. We first introduce a generalization of the measured tree $T(G, \mathcal{X}, \chi)$, discuss the properties of this new structure and then state a characterization of Σ^1 , valid for arbitrary groups. The easy parts of the proof will be given in this section C2.4, the more involved parts in section C2.5.

Notes

C2.4a Notion of a measured tree

In order to discuss quotients of the measured tree $T(G, \mathcal{X}, \chi)$, we introduce the concept of a measured tree, taking the properties of $T(G, \mathcal{X}, \chi)$ enunciated in Lemma C2.23 as axioms of the new structure (T, \leq, λ) . There is a natural notion of morphism for these structures; moreover, the action of a group G by automorphisms on an measured tree (T, \leq, λ) gives rise to a canonical character $\chi_T: G \rightarrow \mathbb{R}$.

DEFINITION C2.34 A *measured tree* is a triple (T, \leq, λ) , consisting of a non-empty set T , an order relation \leq on T and a function $\lambda: T \rightarrow \mathbb{R}$, which satisfies the following axioms:

- (i) (T, \leq) is a tree in the sense of *order theory*; so (T, \leq) is a poset with the additional properties that every finite subset of T has a lower bound and that each subset $(-\infty, a] = \{b \in T \mid b \leq a\}$ is linearly ordered;
- (ii) λ maps each subset of the form $(-\infty, a]$ injectively and in an order-preserving fashion onto a subset of \mathbb{R} that is not bounded from below.

REMARKS C2.35 a) As in the case of the measured tree $T(G, \mathcal{X}, \chi)$, the subset $(-\infty, a]$ occurring in axiom (i) will be called *ray descending from a* .

b) The addendum to axiom (ii), namely that $\lambda((-\infty, a])$ be not bounded from below, is automatically satisfied in all the examples that are of interest in the sequel. This property has therefore been required from the outset, although some results remain valid without it.

The measured tree $T(G, \mathcal{X}, \chi)$, associated to a triple (G, \mathcal{X}, χ) is, of course, an example of a measured tree; other examples are provided by \mathbb{R} -trees with a distinguished end (see example C2.38c). Prior to turning to them, we introduce the notion of morphism of measured trees.

DEFINITION C2.36 Let (T, \leq, λ) and (T', \leq', λ') be measured trees. A *morphism* from the first structure to the second one is a function $\varphi: T \rightarrow T'$ that is order-preserving and for which there exists a real number r such that the equation $\lambda'(\varphi(a)) = \lambda(a) + r$ holds for every point $a \in T$.

In the sequel, three types of morphisms will be important: *automorphisms* induced by groups acting on measured trees, *inclusions* of subtrees and *epimorphisms* of measured trees. We begin with group actions.

PROPOSITION C2.37 Let G be a group which acts by morphisms on a measured tree (T, \leq, λ) . Then the function $g \mapsto \lambda(g.a) - \lambda(a)$ does not depend on the point $a \in T$ and yields a homomorphism $\chi_T: G \rightarrow \mathbb{R}$.

Proof. For every $g \in G$ the function $x \mapsto g.x$ is a morphism of (T, \leq, λ) ; by definition C2.36 there exists therefore a number r such that the equation $\lambda(g.x) =$

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$\lambda(x) + r$ is valid for every $x \in T$. It follows that the function $\chi_T(g) = \lambda(g.a) - \lambda(a)$ does not depend on the choice of a . Assume now that g and h are elements of G , pick $a \in T$ and set $b = h.a$. The computation

$$\begin{aligned}\chi_T(gh) &= \lambda(gh.a) - \lambda(a) \\ &= (\lambda(g.b) - \lambda(b)) + (\lambda(h.a) - \lambda(a)) = \chi_T(g) + \chi_T(h)\end{aligned}$$

then shows that χ_T is a homomorphism of G into the additive group of \mathbb{R} . \square

The homomorphism χ_T provided by the above proposition will be referred to as the *character associated to the action of G on the measured tree T* .

EXAMPLES C2.38 a) Consider the triple $(T = \mathbb{R}, \leq, \lambda = \mathbb{1})$ where \leq denotes the usual order relation on the field of real numbers. This triple is a measured tree. A function of \mathbb{R} into itself is an endomorphism φ of the triple if it is increasing and if there exists a real number r with $\varphi(x) = x + r$ for $x \in \mathbb{R}$; this additional property says that φ is a translation τ with amplitude r . Suppose now G is a group which acts on $(\mathbb{R}, \leq, \mathbb{1})$ by automorphism. Then G acts by translations and the character $\chi_{\mathbb{R}}$ associated to this action is the function which associates to each $g \in G$ the amplitude r_g of the translation $\tau_g: \mathbb{R} \xrightarrow{\sim} \mathbb{R}$.

b) Let $T(G, \mathcal{X}, \chi)$ be the measured tree associated to a group G , a generating system $\eta: \mathcal{X} \rightarrow G$ and a non-zero character χ . By Lemma C2.23, it is a measured tree in the sense of definition C2.34. The group G acts on $T(G, \mathcal{X}, \chi)$ in a canonical way; in view of lemma C2.27 it acts by automorphisms. Equation (C2.18) then shows that the character χ_T , associated to the action by G , is nothing but χ .

c) Let (T, d) be an \mathbb{R} -tree. So (T, d) is a metric space satisfying the following two requirements: ⁸

- For every couple (a, b) of points in T , there exists a unique geodesic arc $\gamma_{a,b}: [0, d(a, b)] \rightarrow T$ that starts in a and ends in b ; let $[a, b]$ denote the image of γ ;
- if the geodesic segments $[a, b]$ and $[b, c]$ have only the end point b in common, then $[a, b] \cup [b, c]$ is the geodesic segment from a to c .

Assume now that (T, d) admits a geodesic ray $(-\infty, b_*]$ of infinite length and let e be the end of T represented by this ray. For any point $a \in T$ there exists then a unique geodesic ray $(-\infty, a]$ with endpoint a that represents e , i. e. which intersects $(-\infty, b_*]$ in a subray. The geodesic rays representing e enable one to define a relation on T , defined by

$$a_1 \leq a_2 \iff (-\infty, a_1] \subseteq (-\infty, a_2]. \quad (\text{C2.20})$$

This relation is an order relation; in addition, it is directed.

⁸See, e. g., [Sha87, p. 276, sections 2.3 and 2.5] or [Chi01], Definition on page 29 and remark at the top of page 30.

The distance function d , on the other hand, allows one to define a function $\lambda: T \rightarrow \mathbb{R}$ as follows: one chooses a base point $b_* \in T$ and sets

$$\lambda(x) = d(c, x) - d(c, b_*) \quad \text{with } c \leq x \text{ and } c \leq b_*. \quad (\text{C2.21})$$

(The properties of an \mathbb{R} -tree imply that $\lambda(x)$ does not depend on the choice of c .) The function λ maps each ray $(-\infty, a]$ bijectively and in a fashion that respects the order relations onto the interval $(-\infty, \lambda(a)]$ of the real line \mathbb{R} .

So far we have verified that the triple (T, \leq, λ) , associated to the \mathbb{R} -tree (T, d) and its end e , is a measured tree in the sense of definition C2.34. Suppose now that a group G acts on (T, d) by isometries which fix the end e ; this assumption amounts to say that G permutes the rays descending from the points of T . We claim that G acts on (T, \leq, λ) by morphisms. Two properties need to be checked: that G acts by automorphisms of the poset (T, \leq) and that the function $\varphi_g: x \mapsto \lambda(g.x) - \lambda(x)$ is constant for each $g \in G$. The first property follows immediately from the definition (C2.20) of the order relation \leq and from the fact that G permutes the rays descending from the points of T .

The verification of the second property needs a short computation. Let b_* denote the selected base point (occurring in definition (C2.21)) and let x be an arbitrary point. Pick a point c in the intersection of the rays descending from the points x , $g.x$ and b_* . Then $c \leq x$, whence $g.c \leq g.x$; thus c and $g.c$ lie both on the ray descending from $g.x$. Since rays are linearly ordered, either $c \leq g.c$, or $g.c < c$. If $c \leq g.c$, then $d(c, g.x) = d(c, g.c) + d(g.c, g.x) = d(c, g.c) + d(c, x)$ and so

$$\begin{aligned} \lambda(g.x) - \lambda(x) &= (d(c, g.x) - d(c, b_*)) - (d(c, x) - d(c, b_*)) \\ &= d(c, g.x) - d(c, x) = d(c, g.c). \end{aligned}$$

If $g.c < c$, one finds similarly that $\lambda(g.x) - \lambda(x) = -d(c, g.c)$. The claim now follows from the fact that the number $\lambda(x)$, defined by equation (C2.21) with the help of b_* and c , does not depend on c .⁹

C2.4b Lines and axes

The characterization of Σ^1 in terms of actions on measured trees uses the concept of a line; in the applications, lines often arise as axes of hyperbolic elements. It is therefore essential that the notion of a line can be defined for all measured trees and that it enjoys the familiar properties.

Definition C2.25 makes sense for all measured trees; we use it to define lines in the more general set-up:

DEFINITION C2.39 Let (T, \leq, λ) be a measured tree. A subset $L \subseteq T$ is called a *line* if it is linearly ordered, contains the ray of each of its elements and if the set $\{\lambda(c) \mid c \in L\}$ is not bounded from above.

⁹The distance $d(c, g.c)$ is the translation length of $g \in G$.

The lines that play a crucial rôle in the sequel are the axes of hyperbolic elements. Their definition relies on that of a characteristic subset. This definition, in turn, can be stated as in definition C2.28:

DEFINITION C2.40 Suppose G acts on a measured tree (T, \leq, λ) by automorphisms. For every $g \in G$, the subset

$$A_g = \{a \in T \mid a \leq g.a \text{ or } g.a \leq a\}. \quad (\text{C2.22})$$

is then called the *characteristic subset* of g .

The salient features of a characteristic subset are collected in the next lemma. It is the analogue of Lemma C2.29 and it can be proved as its model, except for the fact that references to the properties of $T(G, \mathcal{X}, \chi)$ have to be replaced by references to the axioms in Definition C2.34.

LEMMA C2.41 Suppose G acts by automorphism on a measured tree (T, \leq, λ) . Then, for every $g \in G$, the characteristic subset A_g enjoys the following properties:

- (i) A_g is not empty and contains with each $c \in A_g$ the ray descending from c .
- (ii) If $\chi(g) = 0$ then A_g coincides with the set $T^g = \{a \in T \mid g.a = a\}$ of points fixed by g .
- (iii) If $\chi(g) \neq 0$ then A_g is a line. It is the unique $\text{gp}(g)$ -invariant line of T and, for every point $a \in A_g$, the line A_g coincides with the union of the rays descending from the points of the orbit $\text{gp}(g).a$.

REMARK C2.42 An element $g \in G$ will be called *elliptic* if it belongs to the kernel of χ , and otherwise *hyperbolic*. By the previous lemma, the characteristic subset A_g of a hyperbolic element is a line; it will be referred as the *axis* of g .

C2.4c Characterizations of Σ^1

We now come to the main result of section C2.4, a characterization of Σ^1 in terms of actions on measured trees and a second one in terms of actions on \mathbb{R} -trees.

THEOREM C2.43 For every group G and every non-zero character $\chi: G \rightarrow \mathbb{R}$ the following statements are equivalent:

- (i) $[\chi] \in \Sigma^1(G)$;
- (ii) for every generating system $\eta: \mathcal{X} \rightarrow G$ the measured G -tree $T(G, \mathcal{X}, \chi)$ is a line;
- (iii) every measured G -tree with associated character $\chi_T = \chi$ contains a G -invariant line;

Notes

(iv) every (G, \mathbb{R}) -tree (T, d) which satisfies the restrictions

- a) G fixes an end e of (T, d) ,
 - b) the character χ_T associated to this end e coincides with χ ,
- contains a G -invariant line.

Proof. By its very construction, the measured tree $T(G, \mathcal{X}, \chi)$ is a line if, and only if, the subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected. Indeed, $T(G, \mathcal{X}, \chi)$ is a line if, and only if, all the subgraphs $\Gamma(G, \mathcal{X})_\chi^{[\chi(g), \infty)}$ are connected. These various subgraphs are permuted by G and hence isomorphic. Biimplication (i) \Leftrightarrow (ii) is thus valid.

Consider now biimplication (ii) \Leftrightarrow (iii). For every generating system $\eta: \mathcal{X} \rightarrow G$, the measured tree $T(G, \mathcal{X}, \chi)$ is a measured tree in the sense of Definition C2.34 (see example C2.38b). If statement (iii) holds, $T(G, \mathcal{X}, \chi)$ will therefore contain a G -invariant line. As G acts transitively on $T(G, \mathcal{X}, \chi)$, this means that $T(G, \mathcal{X}, \chi)$ is a line. So (iii) implies (ii). The proof of the converse is harder. It relies on the fact that every measured tree (T', \leq', λ') gives rise to a generating system $\eta: \mathcal{X} \rightarrow G$ that permits one to construct a G -equivariant morphism $\varphi: T(G, \mathcal{X}, \chi) \rightarrow (T', \leq', \lambda')$. If $T(G, \mathcal{X}, \chi)$ is a line, its image under this morphism φ will be a G -invariant line of (T', \leq', λ') . Details are given in the proof of Proposition C2.46 below.

We are left with justifying the implications (iii) \Rightarrow (iv) and (iv) \Rightarrow (ii). Let (T, d) be a (G, \mathbb{R}) -tree that satisfies conditions a), b) enunciated in statement (iv). Then the metric space (T, d) gives rise to a measured tree (T, \leq, λ) on which G acts by automorphisms (see example C2.38c for details). As every line in the measured tree (T, \leq, λ) is a geodesic line in the \mathbb{R} -tree (T, d) (see Definition C2.20 of the order relation \leq) and as the action of G on the set T is the same in both structures, statement (iv) is therefore a consequence of statement (iii).

The proof of implication (iv) \Rightarrow (ii) employs another construction. Assume statement (ii) does *not* hold. Then there exists a generating system $\eta: \mathcal{X} \rightarrow G$ so that the associated measured tree $T(G, \mathcal{X}, \chi)$ is *not* a line. The aim is now to fabricate, with the help of $T(G, \mathcal{X}, \chi)$, a (G, \mathbb{R}) -tree (X, d) , and then to verify that this tree does not contain a G -invariant line. The details of this construction will be given in section C2.5b. \square

REMARKS C2.44 a) For *finitely* generated groups G , biimplication (i) \Leftrightarrow (iv) is due to Ken Brown, except for the fact that Brown uses the invariant $\Sigma_{G'}(G)$ discussed in [BNS87] (hence the restriction to finitely generated groups), and right actions (see [Bro87b, p. 499, Corollary 7.4]). Brown's proof involves a characterization of $\Sigma^1(G)$ in terms of so-called *HNN valuations* and divides into two parts: in the first one, he establishes that a point $[\chi]$ lies outside $\Sigma^1(G)$ if, and only if, there is a non-trivial HNN valuation on G with associated character χ (see Theorem 5.2). In the proof of Theorem 7.1 Brown then constructs, given a non-trivial HNN valuation of a group G with associated character χ , a (G, \mathbb{R}) -tree which has

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a fixed end and associated character χ , but contains no G -invariant line. This construction is based on the Alperin-Moss generalization of Chiswell's Theorem 1 in [Chi76] (see, e. g., [AB87, p. 303, Theorem 3.9]).

Both Chiswell and Alperin-Moss start out with a Lyndon length function defined on a group G . This function is intended to measure the distance $d(x_0, g.x_0)$ in the (G, \mathbb{R}) -tree (T, d) between the base point x_0 and its image $g.x_0$ under the action of $g \in G$, once the tree (T, d) and the action of G on it have been found. In Brown's situation the G -action on the tree (T, d) is of a special sort: it fixes an end and G acts by translations along the end.

The question thus arises whether a suitable (G, \mathbb{R}) -tree can be obtained in a more direct manner. This question has been answered in the affirmative, first by Robert Bieri in the late 1980s (cf. [BS92, Chapt. II, proof of Thm. 4.1]) and later by Gilbert Levitt in [Lev94] (see Proposition 3.3). In section C2.5b, I shall give a construction using bits of these solutions.

b) Theorem C2.43 has many useful applications, in particular to joins of subgroups (see section C2.6a), groups with a locally nilpotent normal subgroup (see section C2.6b) and to direct products of groups (see section C2.6c).

C2.5 Construction of morphisms

In this section, two auxiliary results will be established that have been used in the proof of Theorem C2.43. In the first of them, one is given a measured G -tree (T', \leq', λ) whose associated character $\chi: G \rightarrow \mathbb{R}$ is non-zero. One then shows that there exists a set of generators \mathcal{X} and a G -equivariant morphism $\varphi: T(G, \mathcal{X}, \chi) \rightarrow (T', \leq', \lambda')$. The second result starts out with a measured G -tree $T(G, \mathcal{X}, \chi)$ whose character $\chi: G \rightarrow \mathbb{R}$ is non-zero, and constructs a (G, \mathbb{R}) -tree (X, d_X) with associated character χ , all in such a way that (X, d_X) has no G -invariant line if the measured G -tree $T(G, \mathcal{X}, \chi)$ is not a line.

C2.5a Morphism into a measured G -tree

In the proof of Proposition C2.46 the following lemma will be used:

LEMMA C2.45 *Suppose (T, \leq, λ) and (T', \leq', λ') are measured G -trees and G acts transitively on T . Then a G -equivariant morphism $\varphi: (T, \leq, \lambda) \rightarrow (T', \leq', \lambda')$ exists if, and only if, one can find points $b \in T$ and $b' \in T'$ so that the implication*

$$b \leq h.b \implies b' \leq' h.b' \quad (\text{C2.23})$$

holds for every element $h \in G$.

Proof. Assume first a G -equivariant morphism $\varphi: (T, \leq, \lambda) \rightarrow (T', \leq', \lambda')$ exists. Pick $b \in T$ and set $b' = \varphi(b)$. Then implication (C2.23) holds, for it is a consequence of the compatibility of φ with respect to the partial orderings \leq and \leq' required in Definition C2.36.

Notes

Conversely, assume $b \in T$ and $b' \in T'$ are points enjoying property (C2.23). Let S be the stabilizer of the point b . Then the inequalities $b \leq h.b$ and $b \leq h^{-1}.b$ hold for each element $h \in S$ whence S stabilizes b' by implication (C2.23). The assignment $g.b \mapsto g.b'$ is therefore licit; as G acts transitively on T , it defines a map $\varphi: T \rightarrow T'$. Implication (C2.23) also entails that φ is compatible with the order relations \leq and \leq' ; indeed, if $g_1.b \leq g_2.b$ then $b \leq (g_1^{-1}g_2).b$ (for G acts by order preserving automorphisms on (T, \leq, λ)), so $b' \leq (g_1^{-1}g_2).b$ by (C2.23) and thus $g_1.b' \leq g_2.b'$. The computation

$$\begin{aligned} \lambda'(\varphi(g.b)) - \lambda(g.b) &= \lambda'(g.b') - \chi(g.b) = (\chi(g) + \lambda'(b')) - (\chi(g) + \lambda(b)) \\ &= \lambda'(b') - \lambda(b) \end{aligned}$$

finally implies that the difference $\lambda'(\varphi(a)) - \lambda(a)$ does not depend on $a = g.b$ and so φ satisfies also the second property required in Definition C2.36. \square

Now to the construction of a morphism into a given measured G -tree:

PROPOSITION C2.46 *Let G be a group, $\chi: G \rightarrow \mathbb{R}$ a non-zero character and (T', \leq', λ') a measured G -tree with associated character χ . Choose an element $t \in G$ with positive χ -value and a point $b' \in T'$ on the axis of t . Then there exists a generating set $\eta: \mathcal{X} \hookrightarrow G$ so that the function $g \mapsto g.b'$ induces a G -equivariant morphism $\varphi: T(G, \mathcal{X}, \chi) \rightarrow (T', \leq', \lambda')$.*

Proof. We begin by constructing a subset $\mathcal{X} \subset G_\chi$ with the property that the relation $b' \leq x.b'$ holds for every $x \in \mathcal{X}$. Since b' is on the axis of t and as $\chi(t) > 0$, the desired relation holds for $x = t$. Consider now an arbitrary element $z \in G$. The rays descending from $z.b'$ and b' intersect; since $b' \in A_t$ there exists therefore a positive integer m_z with $t^{-m_z}.b' \leq z.b'$, and so $b' \leq (t^{m_z}z).b'$. Set

$$\mathcal{X} = \{t\} \cup \{t^{m_z} \cdot z \mid z \in G\}.$$

Then \mathcal{X} is contained in G_χ and it generates the group G .

Let Γ be the Cayley graph $\Gamma(G, \mathcal{X})$ and let \mathcal{C}_1 denote the connected component of the subgraph Γ_χ containing the unit element 1 of G . We assert that $b' \leq h.b'$ for every group element h in the vertex set of \mathcal{C}_1 . The proof will be by induction on the length ℓ of the $(\mathcal{X} \cup \mathcal{X}^{-1})$ -word $w = y_1 \cdots y_\ell$ that describes a path $p = (1, w)$ from 1 and h . If $\ell = 0$, then $h = 1$ and the claim holds. Assume therefore that $\ell > 0$ and write $h = h' \cdot y_\ell$. Then $b' \leq h'.b'$ by the inductive hypothesis. If $y = y_\ell \in \mathcal{X}$ then $b' \leq y.b'$ by the choice of \mathcal{X} , whence $h'.b' \leq h'.(y.b') = h.b'$ and so $b' \leq h.b'$. If, on the other hand, $y^{-1} \in \mathcal{X}$ then $y.b' \leq b'$ and $h.b' = h'.(y.b') \leq h'.b'$. The points b' and $h.b'$ lie therefore both to the ray descending from $h'.b'$. As this ray is totally ordered and as $\chi(h) \geq 0$, the relation $b' \leq h.b'$ must hold.

The component \mathcal{C}_1 is a point of the measured tree $T(G, \mathcal{X}, \chi)$; call it b . The inequality $b \leq h.b$ holds precisely for the vertices of \mathcal{C}_1 ; in view of the preceding paragraph implication (C2.23) is therefore satisfied. But if so, Lemma C2.45 allows

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one to conclude that the assignment $g.C_1 \mapsto g.b'$ defines a G -equivariant morphism $\varphi: T(G, \mathcal{X}, \chi) \rightarrow (T', \leq', \lambda')$. \square

NOTE C2.47 The proof of Proposition C2.46 goes back to the proof of Theorem 4.1 in Chapter II of [BS92].

C2.5b Morphism into a (G, \mathbb{R}) -tree with fixed end

In this section, we show that a measured tree of the form $T(G, \mathcal{X}, \chi)$ can be “completed” to a (G, \mathbb{R}) -tree (X, d_X) with a fixed end and associated character χ . The details are spelled out in

PROPOSITION C2.48 *Given a group G , a generating system $\eta: \mathcal{X} \rightarrow G$ of G and a non-zero character χ of G , let $T = T(G, \mathcal{X}, \chi)$ be the measured tree associated to the triple (G, \mathcal{X}, χ) . Define a function $d: T \times T \rightarrow \mathbb{R}$ be the formula*

$$d(a_1, a_2) = \inf\{\lambda(a_1) + \lambda(a_2) - 2\lambda(a) \mid a \leq a_1 \text{ and } a \leq a_2\}. \quad (\text{C2.24})$$

Then d is a pseudo-metric on T ; let (\bar{T}, \bar{d}) denote the associated metric space. This metric space can be embedded into a (G, \mathbb{R}) -tree (X, d_X) with the following properties:

- (i) *if the rank $\text{rk}(\chi)$ of χ is 1, then d is a metric, (T, d) is a (G, \mathbb{Z}) -tree and (X, d_X) can be chosen to be the geometric realization of (T, d) .*
- (ii) *if $\text{rk}(\chi) > 1$, there exists a G -equivariant isometry $\varphi: (T, d) \rightarrow (X, d_X)$ with dense image;*
- (iii) *if the (G, \mathbb{R}) -tree (X, d_X) contains a G -invariant line the measured tree $T = T(G, \mathcal{X}, \chi)$ is a line and the subgraph $\Gamma(G, \mathcal{X})_\chi$ is connected.*

Proof. Let $d: T \times T \rightarrow \mathbb{R}$ be the function defined by formula (C2.24). Since every pair of points $\{a_1, a_2\}$ in T is preceded by some element $a \in T$, the definition is licit. The function d is symmetric and vanishes if $a_1 = a_2$; moreover, a short calculation will confirm that it satisfies the triangle inequality, and so it is a pseudo-metric. The computation

$$\begin{aligned} d(g.a_1, g.a_2) &= \inf\{\lambda(g.a_1) + \lambda(g.a_2) - 2\lambda(b) \mid b \leq g.a_1 \text{ and } b \leq g.a_2\} \\ &= \inf\{\lambda(g.a_1) + \lambda(g.a_2) - 2\lambda(g.a) \mid a \leq a_1 \text{ and } a \leq a_2\} \\ &= \inf\{\lambda(a_1) + \lambda(a_2) - 2\lambda(a) \mid a \leq a_1 \text{ and } a \leq a_2\} \\ &= d(a_1, a_2) \end{aligned}$$

finally proves that this pseudo-metric is G -invariant.

Let (\bar{T}, \bar{d}) be the resulting metric space; as the pseudo-metric d is G -invariant, so is the metric \bar{d} . Let $\pi: T \rightarrow \bar{T}$ denote the canonical projection; it sends the point a to the ball \bar{a} of radius 0 centered at a , and it is G -equivariant.

Notes

Claim (i) If $\text{im } \chi$ is infinite cyclic, we may and shall assume that $\text{im } \chi = \mathbb{Z}$. The function λ maps then T onto \mathbb{Z} . For every $j \in \mathbb{Z}$, let $V_j = \lambda^{-1}(\{j\})$ denote the set of components of the subgraph $\Gamma_\chi^{[j, \infty)}$. Then $T = \bigcup_{j \in \mathbb{Z}} V_j$. Moreover, every vertex $v \in V_j$ is preceded by a unique $w_v \in V_{j-1}$ and so gives rise to a positively oriented edge $e_v = (v, w_v)$; let E^+ be the set of all these positive edges and set $E = E^+ \cup E^-$ (cf. section A1.2a). Then (T, E) is a \mathbb{Z} -tree and the action of G on T turns it into a (G, \mathbb{Z}) -tree. It can be embedded into an (G, \mathbb{R}) -tree by replacing the oriented edges (v, w_v) by segments $[v, w_v]$ that are isometric to the unit interval $[0, 1] \subset \mathbb{R}$ and extending the metric and the action suitably (see, e. g., [AB87, Cor. 4.5], [BH99, Section 1.9], or [Chi01, pp. 44–49]).

Claim (ii) If the rank of χ is bigger than 1, the image of χ is a dense subgroup of \mathbb{R}_{add} ; the space (\bar{T}, \bar{d}) , however, need not be path-connected and so it may not be an \mathbb{R} -tree. We therefore add to \bar{T} certain points of its metric completion. One way of controlling these additions consists in resorting to Chiswell's construction of a rooted tree (see, e. g., [AB87, pp. 299–308]). We shall reach our goal by a different route: we shall equip \bar{T} with the structure of a measured G -tree, use this structure to define a subspace (X, d_X) of the metric completion of (\bar{T}, \bar{d}) and then verify that (X, d_X) carries a canonical G -action and is actually a (G, \mathbb{R}) -tree.

Step 1: construction of the measured G -tree $(\bar{T}, \leq', \lambda')$. We introduce a relation \leq' on \bar{T} and a function $\lambda': \bar{T} \rightarrow \mathbb{R}$ by setting:

$$\bar{a}_1 \leq' \bar{a}_2 \iff \text{there exists } b \in T \text{ with } d(a_1, b) = 0 \text{ and } b \leq a_2, \quad (\text{C2.25})$$

$$\lambda'(\bar{a}) = \lambda(a). \quad (\text{C2.26})$$

The definition of λ' is licit because the function λ is constant on each ball of radius 0. Indeed, suppose c_1 and c_2 are at distance 0. For every positive real number ε there exists then a point $c \in T$ which precedes both c_1 and c_2 and satisfies the inequality

$$(\lambda(c_1) - \lambda(c)) + (\lambda(c_2) - \lambda(c)) < \varepsilon.$$

This inequality implies that $|\lambda(c_1) - \lambda(c_2)|$ is bounded by ε .

The definition of \leq' is also licit. Indeed, suppose \tilde{a}_1 and \tilde{a}_2 are points with $d(\tilde{a}_1, a_1) = d(\tilde{a}_2, a_2) = 0$. Then $d(\tilde{a}_1, b) = 0$ and two cases arise: if $\lambda(b) < \lambda(a_2)$, the definition of the pseudo metric d implies that the rays descending from a_2 and from \tilde{a}_2 both contain b whence $b \leq \tilde{a}_2$. So suppose that $\lambda(b) = \lambda(a_2)$. Then the points b and a_2 coincide and so $d(b, \tilde{a}_2) = 0$, whence $d(\tilde{a}_1, \tilde{a}_2) = 0$. The rôle of the auxiliary element b occurring in equation (C2.25) can therefore be played by \tilde{a}_2 .

We verify next that \leq' is an order relation. It is clear that the relation is reflexive. Suppose that $\bar{a}_1 \leq' \bar{a}_2$ and let $b \in T$ be an element with $d(a_1, b) = 0$ and $b \leq a_2$. Then $\lambda(a_1) \leq \lambda(a_2)$. If one has also $\bar{a}_2 \leq' \bar{a}_1$ then $\lambda(a_1) = \lambda(a_2)$, hence $b = a_2$ and $\bar{a}_1 = \bar{a}_2$. To see that the relation is transitive, assume that $\bar{a}_1 \leq' \bar{a}_2$ and $\bar{a}_2 \leq' \bar{a}_3$. There exist then points b and c such that $d(a_1, b) = 0$, $b \leq a_2$ and $d(a_2, c) = 0$, $c \leq a_3$. Two cases arise: if $b = a_2$ then $d(a_1, c) = 0$ and so $\bar{a}_1 \leq' \bar{a}_3$; otherwise, $\lambda(b) < \lambda(a_2)$ whence b lies on the ray descending from a_3

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and so $b \leq a_3$. In both cases, $\bar{a}_1 \leq' \bar{a}_3$. All taken together this proves that \leq' is an order relation.

The fact that (T, \leq) is directed and definition (C2.25) immediately imply that (\bar{T}, \leq') is directed. In addition, the fact that λ maps each ray $(-\infty, a]$ isometrically into the interval $(-\infty, \lambda(a)] \subset \mathbb{R}$ and the definition of \leq' guarantee that λ' has the analogous property. The triple $(\bar{T}, \leq', \lambda')$ is therefore a measured tree; the group G acts on it by automorphisms, for it acts by automorphisms on (T, \leq, λ) . It follows that the canonical projection $\pi: T \rightarrow \bar{T}$ is a G -equivariant morphism of measured G -trees.

Step 2: constructions of the metric space (X, d_X) and of the measured G -tree (X, \leq_X, λ_X) . The metric space (\bar{T}, \bar{d}) can be embedded into a complete metric space, say (Y, d_Y) . This space is too large for our purpose, and so we define a subspace (X, d_X) in which only the rays $(-\infty, \bar{a}]$ of (\bar{T}, \bar{d}) are “completed”.

Let X be the subset made up of the points $y \in Y$ that can be represented by a *descending Cauchy-sequences*, i. e., by Cauchy-sequences

$$n \mapsto \bar{a}_n \quad \text{with} \quad \bar{a}_{n+1} \leq' \bar{a}_n \quad \text{for all } n \in \mathbb{N}. \quad (\text{C2.27})$$

The metric \bar{d} on \bar{T} extends uniquely to a metric d_X on X .

The next aim is to equip the metric space (X, d_X) with the structure of a measured tree. The definition of the measuring function $\lambda_X: X \rightarrow \mathbb{R}$ is straightforward: the function $\lambda': \bar{T} \rightarrow \mathbb{R}$ is Lipschitz-continuous (with constant 1) and so it extends uniquely to a function λ_Y on the complete metric space Y ; let $\lambda_X: X \rightarrow \mathbb{R}$ be its restriction to X . Consider now the ray $\bar{R}_{\bar{a}} = (-\infty, \bar{a}]$ descending from a point $\bar{a} \in \bar{T}$. The function λ' maps it bijectively onto the subset $(-\infty, \lambda'(\bar{a})] \cap \text{im } \lambda$ of the real line \mathbb{R} . Let $R_{\bar{a}} \subset X$ be the subset containing all the limit points of sequences in $\bar{R}_{\bar{a}}$. The extended function $\lambda_X: X \rightarrow \mathbb{R}$ maps $R_{\bar{a}}$ bijectively and isometrically onto the interval $(-\infty, \lambda_X(\bar{a})]$ of \mathbb{R} . We use this bijection to pull back the order relation on the interval $(-\infty, \lambda_X(\bar{a})]$ onto the “ray” $R_{\bar{a}}$ and to obtain thus an order relation \leq_X on $R_{\bar{a}}$. This order relation extends the original order relation on the ray $\bar{R}_{\bar{a}}$ and it does not change on $R_{\bar{a}}$ if the upper limit \bar{a} is replaced by a point $\bar{b} \in \bar{T}$ with $\bar{a} \leq' \bar{b}$. The union of the order relations on all the “rays” $R_{\bar{c}}$ with $\bar{c} \in \bar{T}$ is therefore an order relation on X ; we denote it by \leq_X .

For future reference we summarize the above reasonings in

LEMMA C2.49 *Every ray $(-\infty, x]$ descending from a point $x \in X$ is linearly ordered and λ_X maps it isometrically and in an order-preserving way onto the interval $(-\infty, \lambda_X(x)]$ of \mathbb{R} .*

The triple (X, \leq_X, λ_X) is thus a measured tree. The facts that G acts by isometries on the metric space (\bar{T}, \bar{d}) and permutes the rays of the measured tree $(\bar{T}, \leq', \lambda')$, finally, implies that G acts by isometries on the metric space (X, d_X) and by morphisms on the measured tree (X, \leq_X, λ_X) . Moreover, the inclusion $\mu: \bar{T} \hookrightarrow X$ is a G -equivariant morphism of measured G -trees.

Step 3: verification that (X, d_X) is a (G, \mathbb{R}) -tree. By Steps 1 and 2 we know that the composite $\varphi = \mu \circ \pi$ is a G -equivariant morphism of the measured G -tree

Notes

$T(G, \mathcal{X}, \chi)$ into the measured G -tree (X, \leq_X, λ_X) and that X is equipped with a metric d_X . This metric satisfies the equation

$$d_X(x_1, x_2) = \inf\{\lambda_X(x_1) + \lambda_X(x_2) - 2\lambda_X(x) \mid x \leq_X x_1 \text{ and } x \leq_X x_2\}.$$

To verify that (X, d_X) is an \mathbb{R} -tree, we have to *check two properties*: given points x_1, x_2 of X there must exist a *unique geodesic segment* $[x_1, x_2]$ from x_1 to x_2 , and given three points x_1, x_2 and x_3 such that the geodesic segments $[x_1, x_2]$ and $[x_2, x_3]$ have only the point x_2 in common, *the union* $[x_1, x_2] \cup [x_2, x_3]$ *must be the geodesic segment from* x_1 *to* x_3 (see, e. g., [Sha87, p. 276, items 2.3 and 2.5] or [AB87, Definition 2.9]).

Let x_1 and x_2 be points of X . For the proof of the *existence* of a geodesic arc from x_1 to x_2 , two cases will be distinguished: if $x_1 \leq_X x_2$ or $x_2 \leq_X x_1$, the claim follows directly from Lemma C2.49; otherwise, the local compactness of the rays implies that there exists a point x satisfying the properties

$$x \leq_X x_1, \quad x \leq_X x_2 \text{ and } d_X(x_1, x_2) = (\lambda_X(x_1) - \lambda_X(x)) + (\lambda_X(x_2) - \lambda_X(x)).$$

Set $r_1 = d_X(x, x_1)$. Lemma C2.49 then permits one to define a path

$$\gamma: [0, d_X(x_1, x_2)] \rightarrow (\infty, x_1] \cup (\infty, x_2],$$

given by the formula

$$\gamma(r) = \begin{cases} \text{point } z \text{ on } (\infty, x_1] \text{ with } \lambda(x_1) - \lambda(z) = r & \text{if } r \leq r_1, \\ \text{point } z \text{ on } (\infty, x_2] \text{ with } \lambda(z) - \lambda(x) = r - r_1 & \text{if } r > r_1. \end{cases} \quad (\text{C2.28})$$

This path is a geodesic arc (cf. [Chi01, Lemma 2.1.12]).

We next show that the geodesic arc between two points is *unique*. The strategy will be similar to that used in the proof of Lemma 2.1.12 in Chiswell's monograph [Chi01]. Let $[x_1, x_2]$ denote the image of the previously constructed geodesic arc γ from x_1 to x_2 . This set contains for every real $r \in [0, d_X(x_1, x_2)]$ a unique point x with $d_X(x_1, x) = r$. Consider now an arbitrary geodesic arc γ' from x_1 to x_2 . If its image lies in $[x_1, x_2]$, the geodesic arc γ' must coincide with γ . Otherwise, it contains a point z outside $[x_1, x_2]$. Two cases then arise.

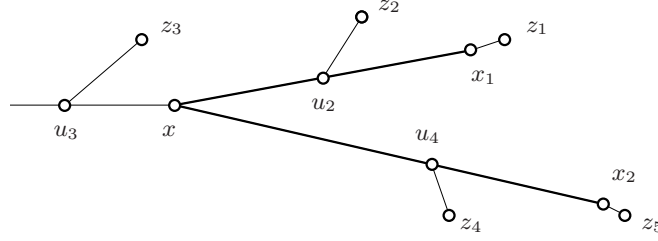
If x_1 and x_2 are *comparable*, we may assume that $x_2 \leq_X x_1$. The rays descending from x_1 and z , respectively, intersect then in a ray $(-\infty, u]$. If $x_1 = u$, then $x_1 <_X z$ and so

$$d_X(x_1, x_2) = d_X(x_1, z) + d_X(z, x_2) = 2d_X(x_1, z) + d_X(x_1, x_2),$$

a contradiction. If $u < x_2$ one arrives at a similar contradictory situation. If, finally, $x_2 \leq_X u \leq_X x_1$, one deduces the equality $d_X(x_1, x_2) = 2d_X(u, z) + d_X(x_1, x_2)$, a third contradiction.

Assume now that x_1 and x_2 are *not comparable* and let x be the point where the rays descending from x_1 , respectively x_2 , meet. Various cases arise; they are

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Figure C.1: Proof of the uniqueness of the geodesics in (X, d_X)

summarized in Figure C.1. Each of these cases leads to a contradiction in the manner exemplified before. All taken together, we have thus shown that there is a unique geodesic arc from a given point x_1 to a given point x_2 .

We are left with the verification of the second characteristic property. Let x_1, x_2, x_3 be three points of X and let $[x_1, x_2]$ denote the geodesic segment with endpoints x_1, x_2 , and $[x_2, x_3]$ the segment with endpoints x_2, x_3 . Suppose these segments have only the point x_2 in common; we claim that $[x_1, x_2] \cup [x_2, x_3]$ is a geodesic segment. This is clearly the case if $x_2 \leq_X x_1$ or if $x_2 \leq_X x_3$. If, on the other hand, neither $x_2 \not\leq_X x_1$ nor $x_2 \not\leq_X x_3$ were valid, there would exist a point $x <_X x_2$ so that the segment $[x, x_2]$ would be contained both in $[x_1, x_2]$ and in $[x_2, x_3]$, a flagrant contradiction to the assumption that the segments $[x_1, x_2]$ and $[x_2, x_3]$ meet only in the point x_2 .

Claim (iii) Assume first that $\text{rk } \chi > 1$; then $\text{im } \chi$ is a dense subset of \mathbb{R} . Let L be a G -invariant line in the (G, \mathbb{R}) -tree (X, d_X) . Our first aim is to prove that L is a linearly ordered subset of the measured tree (X, \leq_X, λ_X) .

Fix an element $h \in H$ with $\chi(h) > 0$. Since h is an automorphism of the measured tree (X, \leq_X, λ_X) , the equation

$$\lambda_X(h.y) = \chi(h) + \lambda_X(y) > \lambda_X(y) \quad (\text{C2.29})$$

holds for every point $y \in X$. This equation shows that h has no fixed points and so the isometry α_h induced by h on the (G, \mathbb{R}) -tree (X, d_X) is hyperbolic (cf. [Chi01, p. 82]). The axis L_h of h is a geodesic line; it coincides with L and can be written as a union of segments

$$\bigcup_{j \in \mathbb{Z}} [h^j.x, h^{j+1}.x];$$

here x is an arbitrary point of L_h (cf. [Chi01, Thm. 3.1.4]). Consider now the segment $[x, h.x]$ of X . According to Step 3 in the construction of (X, d_X) , this segment contains a point z so that both subsegments $[x, z]$ and $[z, h.x]$ are linearly ordered. If z is the endpoint $h.x$, then x lies on the ray $(-\infty, h.x]$ descending from $h.x$. Otherwise, there would exist an element $z \in X$ with $z \leq_X x$ and $<_X h.x$, hence $h.z \leq_X h.x$. So z and $h.z$ would both belong to the ray descending from $h.x$. Equation (C2.18) with $y = z$ would then show that $z < h.z$, and so the

Notes

computation

$$d_X(z, h.z) < d_X(x, z) + d_X(z, h.z) + d_X(h.z, h.x) = d_X(x, h.x)$$

would be valid; it would contradict the fact that h acts on its axis by a translation.

The above argument shows that $x \leq_X h.x$ for every $x \in L$. So L is contained in the axis A_h of h in the measured tree (X, \leq_X, λ_X) ; since L contains the orbit $\text{gp}(h).x$ for every $x \in L$, the line L actually coincides with the axis A_h of h . Lemma C2.49 next shows that the function λ_X maps the axis A_h bijectively onto \mathbb{R} and so L contains points of the measured subtree $(\bar{T}, \leq', \lambda')$. Since \bar{T} is a G -invariant subset of X , the intersection $\bar{A}_h = \bar{T} \cap A_h$ is G -invariant and it is the axis of the action of h on \bar{T} . As G acts transitively on the tree T , hence on the quotient \bar{T} of T , the tree \bar{T} coincides therefore with the axis \bar{A}_h of h , and so it is a line.

But if so, *the tree T must be a line*. Indeed: the axis A_h of h in the measured tree $T = T(G, \mathcal{X}, \chi)$ is a line and every point $a_1 \in T$ is at distance 0 from some point $a_2 \in A_h$ (this follows from the definition of \bar{T} and the equality $\bar{T} = \bar{A}_h$). The definition (C2.24) of the pseudo metric d then implies that $h^{-1}.a_1$ and $h^{-1}.a_2$ lie on the ray descending from a_2 and so they coincide. So $a_1 = h.(h^{-1}(a_1)) = a_2 \in A_h$.

Assume now that $\text{rk } \chi = 1$. We may and shall assume that $\text{im } \chi = \mathbb{Z}$. The measured tree $T = T(G, \mathcal{X}, \chi)$ can then be viewed as a \mathbb{Z} -tree, with each edge (v, w) of length 1 and $\lambda(v) = \lambda(w) + 1$ (cf. proof of claim (i)). The tree T is a subtree of the (G, \mathbb{R}) -tree (X, d_X) and the definition of the metric d_X implies that every isometric embedding $\varphi: R \rightarrow X$ passes through a point of this subtree T .

Let L be a G -invariant line of (X, d_X) . Then L is the image of an isometric embedding $\varphi: R \rightarrow X$; by the previous remark it thus contains a point $a_0 \in T$. Choose $t \in G$ with $\chi(t) = 1$. It then follows as in the previous part that t acts on (X, d_X) by a hyperbolic isometry and that it acts on the line L by a translation, say with amplitude α . Since $t.a_0 \in T$ this amplitude is a positive integer. It is actually 1, for otherwise the segment $[a_0, t.a_0] \subset X$ would contain more than two elements of T and one could find, as before, an element $z \in [a_0, t.a_0]$ with $d_X(z, t.z) < c$. The segment $[a_0, h.a_0]$ is therefore the geometric realization of an edge of the \mathbb{Z} -graph $T = T(G, \mathcal{X}, \chi)$. It follows, first, that the line L is linearly ordered and then that the intersection $T \cap L$ is a non-empty, linearly ordered G -invariant subset of T . Since G acts transitively on T this means that T is linearly ordered and thus a line. \square

C2.6 Some applications

Theorem C2.43 is a very powerful tool that allows one to establish, by pleasant geometric arguments, that a point $[\chi]$ belongs to the invariant $\Sigma^1(G)$ of the group under consideration. In the section, this assertion will be illustrated by some examples.

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C2.6a Application 1: joins of subgroups

We begin with a generalization of Proposition B1.15.

PROPOSITION C2.50 *Assume G is generated by a (non-empty) collection of subgroups $\{G_v \mid v \in V\}$. Given a non-zero character $\chi: G \rightarrow \mathbb{R}$, let $\mathcal{G}(\chi)$ denote the combinatorial graph with vertex set V and edge set the set of those pairs $\{u, v\}$ for which χ is non-zero on the intersection $G_u \cap G_v$. If the conditions*

(i) *for every $v \in V$, the restriction of χ to G_v is non-zero and represents a point of $\Sigma^1(G_v)$,*

(ii) *the graph $\mathcal{G}(\chi)$ is connected*

are satisfied, then χ represents a point of $\Sigma^1(G)$.

Proof. Assume conditions (i) and (ii) are satisfied. If V has only one element, then $G = G_v$ and so the claim holds by assumption (i); so suppose $\text{card}(V) > 1$ and consider a measured G -tree (T, \leq, λ) with associated character χ . Our aim is to find a G -invariant line in this tree. By restricting the action of G on (T, \leq, λ) to that of a subgroup H one obtains a measured H -tree. If this observation is applied to one of the subgroups G_v with $v \in V$, hypothesis (i) implies that the restriction χ_v of χ to G_v is non-zero and $[\chi_v] \in G_v$; by Theorem C2.43 the measured G_v -tree T contains therefore an invariant line L_v . This line coincides with the axis of every hyperbolic element $h_v \in G_v$ (cf. claim (iii) in Lemma C2.41).

Consider next an edge $\{u, v\}$ in the graph $\mathcal{G}(\chi)$. Then $G_u \cap G_v$ will contain a hyperbolic element h_{uv} ; let A_{uv} be its axis. As $L_u = A_{uv}$, the axis A_{uv} is G_u invariant; since $L_v = A_{uv}$, the axis A_{uv} is G_v invariant, too, and so it is invariant under the join $\text{gp}(G_u \cup G_v)$ of these two subgroups.

Fix now a vertex v_0 and let L_0 an invariant line of G_{v_0} . The hypotheses that $\mathcal{G}(\chi)$ be connected and an obvious induction then allow one to see that L_0 is invariant under every subgroup G_v with $v \in V$ and so under G .

As the previous argument is valid for every measured G -tree, implication (iii) \Rightarrow (i) of Theorem C2.43 applies and shows that $[\chi] \in \Sigma^1(G)$. \square

REMARKS C2.51 a) In the above proof, we considered an arbitrary measured G -tree; one could equally well have used the measured trees $T(G, \mathcal{X}, \chi)$ associated to (G, \mathcal{X}, χ) with \mathcal{X} ranging over the generating sets of G .

b) The previous proof is phrased in terms of measured G -trees. One could also use (G, \mathbb{R}) -trees with a fixed end and associated character χ . Whether one type of tree or the other is employed seems to be merely a matter of taste; indeed, I know of no instance where a proof that a character χ represents a point of Σ^1 which is based on actions on one type of tree cannot be transformed in a straightforward manner into a proof using actions on the other type of tree.

Notes

Things can be different if one uses Proposition C2.50 to establish that a given character does *not* represent a point in Σ^1 ; then a tree without an invariant line of only one type may be close at hand.

C2.6b Application 2: subnormal subgroups

Our second result generalizes Lemma C1.20:

PROPOSITION C2.52 *Let N be a subnormal subgroup of the group G and let $\chi: G \rightarrow \mathbb{R}$ be a character that does not vanish on N . Assume the restriction $\chi|_N$ of χ represents a point of $\Sigma^1(N)$. Then $[\chi] \in \Sigma^1(G)$.*

Proof. Let (T, \leq, λ) be a measured G -tree with character χ . View it as a measured N -tree by restricting the action. In view of Theorem C2.43 and the hypothesis that $[\chi|_N] \in \Sigma^1(N)$, the measured N -tree T contains an N -invariant line L . Moreover, if $g \in G$ is an element which normalizes N then $g.L$ is invariant under gNg^{-1} . As $gNg^{-1} = N$, this shows that L is invariant under the normalizer of N .

If the previous argument is applied to successive members of a chain $N = N_0 < N_1 < \dots < N_\ell = G$ of subgroups in G which testifies that N is a subnormal subgroup, one sees first that L is invariant under N_1 , then under N_2 , and finally under G . Implication (iii) \Rightarrow (i) of Theorem C2.43, then permits one to conclude that $[\chi] \in \Sigma^1(G)$. \square

COROLLARY C2.53 *Let N be a subnormal subgroup of the group G . If $\Sigma^1(N) = S(N)$ then $\Sigma^1(G)$ contains the complement of the subsphere $S(G, N)$.*

REMARKS C2.54 a) Proposition C2.52 and its corollary are taken from the monograph [BS92] (see section II.4.3).

b) The hypotheses that $\Sigma^1(N)$ be the entire sphere $S(N)$ holds, in particular, if N is nilpotent or, more generally, *locally nilpotent*. This follows, e. g., from Corollary C2.5 and Remark C2.7b; it can also be derived from the fact that the derived group of a finitely generated nilpotent subgroup is finitely generated and Proposition C2.50. We thus see that Corollary C2.53 generalizes lemma C1.20 (in that Lemma, N is assumed to be an abelian normal subgroup of G).

C2.6c Application 3: Σ^1 of a direct product

As a third application, we establish a formula for the complement of the invariant of a direct product of arbitrary groups; in section A2.3b the same formula has already been shown to hold for direct products of finitely generated groups.

PROPOSITION C2.55 *Let $G = G_1 \times G_2$ be the direct product of two arbitrary groups and let π_1, π_2 denote the canonical projections onto the factors. Then*

$$\Sigma^1(G_1 \times G_2)^c = \pi_1^*(\Sigma^1(G_1))^c \cup \pi_2^*(\Sigma^1(G_2))^c. \quad (\text{C2.30})$$

on the Sigma invariants

Proof. Let $\chi = (\chi_1, \chi_2): G_1 \times G_2 \rightarrow \mathbb{R}$ be a non-zero character. Assume first that the restrictions $\chi_i: G_i \rightarrow \mathbb{R}$ are both non-zero and let $g_1 \in G_1$ and $g_2 \in G_2$ be elements with non-zero χ -value. Consider now a measured G -tree (T, \leq, λ) and let A_1 be the axis of the hyperbolic element g_1 . Since g_1 is centralized by G_2 , Lemma C2.31 shows that the axis is G_2 -invariant; moreover, since $\chi(g_2) \neq 0$ this lemma tells us also that A_1 coincides with the axis A_2 of g_2 . But if so, one sees, by exchanging the rôles of G_1 and G_2 in the preceding reasoning, that A_2 is a G_1 -invariant line, whence $A_1 = A_2$ is G -invariant. Use now implication (iii) \Rightarrow (i) of Theorem C2.43.

Suppose next that χ vanishes on one of the factors of $G = G_1 \times G_2$, say on G_1 . We assert that $[\chi]$ lies in $\Sigma^1(G)$ if, and only, if $[\chi_2] \in \Sigma^1(G_2)$. Assume first that $[\chi_2] \in \Sigma^1(G_2)$ and consider a measured G -tree (T, \leq, λ) . Since G_2 is a subgroup of G the tree can be viewed as a measured G_2 -tree. By implication (i) \Rightarrow (iii) of Theorem C2.43 this tree has a G_2 -invariant line; it is the axis of A_2 of a hyperbolic element $g_2 \in G_2$. As g_2 is centralized by G_1 , Lemma C2.31 next shows that A_2 is G_2 -invariant. But if so, this axis is $G = \text{gp}(G_1 \cup G_2)$ -invariant; as (T, \leq, λ) is an arbitrary measured G -tree, implication (iii) \Rightarrow (i) of Theorem C2.43 thus shows that $[\chi] \in \Sigma^1(G)$. Conversely, assume that $[\chi_2] \in \Sigma^1(G_2)^c$. By Theorem C2.43 there exists then a measured G_2 -tree (T, \leq, λ) that contains no G_2 -invariant line. View this tree as a G -tree by pulling back the action along $\pi_2: G \twoheadrightarrow G_2$. The resulting G -tree has no G -invariant line, whence implication (i) \Rightarrow (iii) of Theorem C2.43 allows one to conclude that $[\chi] \in \Sigma^1(G)^c$. \square

C3 Variation adapted to fundamental groups

In this section, we present an alternate description of the invariant for groups that are fundamental groups of finite connected CW-complexes Y or of compact, connected manifolds. Given a non-zero character $\chi: G \rightarrow \mathbb{R}$, the requirement that the subgraph Γ_χ of a Cayley graph $\Gamma(G, \mathcal{X})$ be connected will be replaced by a condition that involves a function $f: X \rightarrow \mathbb{R}$ and the subspace $f^{-1}([0, \infty))$. Here $X = \hat{Y}$ is the *universal abelian cover* of the space Y and f denotes a continuous function satisfying the equation

$$f(g.x) = f(x) + \chi(g). \quad (\text{C3.1})$$

The formula says that f is *G-equivariant*: it is compatible with the action on X given by the deck transformations of the covering projection $p: X \twoheadrightarrow Y$ and the action by the translations $r \mapsto r + \chi(g)$ on the real line. (TO BE COMPLETED)

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